Denseness of $C_0^\infty (\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p}(\mathbb{R}^n)$

by

Stefan Samko

1. Introduction

The spaces $L^p(x)(\Omega), \Omega \subseteq \mathbb{R}^n$, with variable order $p(x)$ were studied recently. We refer to the pioneer work by I.I. Sharapudinov [6] and the later papers by O.Kováčik and J. Rákosník [2] and by the author [3]-[5]. In the paper [2] the Sobolev type spaces $W^{m,p}(\Omega)$ were also studied. D.E.Edmunds and J. Rákosník [1] dealt with the problem of denseness of $C^\infty$-functions in $W^{m,p}(\Omega)$ and proved this denseness under some special monotonicity-type condition on $p(x)$. We prove that $C_0^\infty (\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$ without any monotonicity condition, requiring instead that $p(x)$ is somewhat better than just continuous - satisfies the Dini-Lipschitz condition. For this purpose we prove the boundedness of the convolution operators $\frac{1}{\epsilon^n} K (\frac{x}{\epsilon}) * f$ in the space $L^p(x)$ uniform with respect to $\epsilon$. This is the main result, the above mentioned denseness being its consequence, in fact.

In the one dimensional periodical case a similar result for the uniform boundedness in $L^p(x)$ of some family of operators $K_\epsilon$, depending on $\epsilon$, was proved by I.I.Sharapudinov [7].

2. Preliminaries

We refer to the papers [2]-[6] for basics of the spaces $L^p(x)$, but remind their definition and some important properties.

Let $p(x)$ be a measurable function on a domain $\Omega \subseteq \mathbb{R}^n$ satisfying the condition $1 \leq p(x) \leq \infty$ and let

$$E_\infty = E_\infty (p) = \{ x \in \Omega : p(x) = \infty \}.$$  

We denote

$$P = \sup_{x \in \Omega \setminus E_\infty (p)} p(x) , \ p_0 = \inf_{x \in \Omega} p(x).$$

where sup and inf stand for esssup and essinf, respectively. By $L^p(x)(\Omega)$ we denote the space of measurable functions $f(x)$ on $\Omega$ such that

$$I_p(f) := \int_{\Omega \setminus E_\infty} |f(x)|^{p(x)} \, dx < \infty \quad \text{and} \quad f(x) \in L^\infty(E_\infty).$$

Let

$$\|f\|_p = \inf \left\{ \lambda > 0 : I_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}.$$  

(1)
In case of $P < \infty$ the space $L^{p(x)}$ is a Banach space with respect to the norm
\[ \| f \|_p = \| f \|_{(p)} + \| f \|_{L^\infty(E_{\infty})}. \]  

We emphasize that $\| f \|_p$ is finite for any $f(x) \in L^{p(x)}(\Omega)$ in the case $P = \infty$ as well, but $L^{p(x)}(\Omega)$ is not a linear space and $\| f \|_p$ is not a norm in this case.

We note the following properties of the space $L^{p(x)}(\Omega)$:

a) the Hölder inequality ([6],[2],[3]) :
\[ \int_{\Omega} |f(x)\varphi(x)| \, dx \leq k\| f \|_p\| \varphi \|_q \]  

where $1 \leq p(x) \leq \infty$, $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, $k = \sup_{x \in \Omega} \| f \|_{p(x)} + \sup_{x \in \Omega} \frac{1}{q(x)}$;

b) inequalities between $I_p(f)$ and $\| f \|_{(p)}$ ([6],[2],[3]) :
\[ \| f \|_p^{(p)} \leq I_p(f) \leq \| f \|_{(p)}^{(p)} , \quad \text{if} \quad \| f \|_{(p)} \leq 1 \]  
\[ \| f \|_{(p)}^{(p)} \leq I_p(f) \leq \| f \|_p^{(p)} , \quad \text{if} \quad \| f \|_{(p)} \geq 1 \]  

the left-hand side inequality in (4) and the right-hand side one in (5) being trivial in the case $P = \infty$ ;

c) estimates for the norm of the characteristic function of a set ([3]) :
\[ |E|^{\frac{1}{p}} \leq \| \chi_E \|_{(p)} \leq |E|^{\frac{1}{p_0}} , \quad \text{if} \quad |E| \leq 1 , \quad E \subseteq \Omega \setminus E_{\infty}(p), \]  

the signs of the inequalities being opposite if $|E| \geq 1$; here $|E|$ is the Lebesgue measure of $E$ ; as in (4)-(5), the corresponding inequalities are trivial in the case $P = \infty$ ;

d) the embedding theorem ([3]) : let $1 \leq r(x) \leq p(x) \leq P < \infty$ for $x \in \Omega$ and $|\Omega| < \infty$. Then $L^{p(x)} \subseteq L^{r(x)}$ and
\[ \| f \|_r \leq (a_2 + (1 - a_1)|\Omega|)\| f \|_p \]  

where $a_1 = \inf_{\Omega} \frac{r(x)}{p(x)}$, $a_2 = \sup_{\Omega} \frac{r(x)}{p(x)}$, see also [2] for this imbedding without the restriction $p(x) \leq P < \infty$ , but with worse constants $a_2 = 1$ and $1 - a_1 = 1$ .

e) denseness of step functions ([3]) : functions of the form $\sum_{k=1}^m c_k \chi_{\Omega_k}, \Omega_k \subset \Omega, |\Omega_k| < \infty$, with constant $c_k$, form a dense set in $L^{p(x)}(\Omega)$.

As in [4]-[5], we use the weak Lipschits condition (Dini-Lipschits condition):
\[ |p(x) - p(y)| \leq A \log \frac{1}{|x-y|} , \quad |x - y| \leq \frac{1}{2}. \]  

Everywhere below we assume that $P < \infty$.

3. Statements of the main results

Let $\mathcal{K}(x)$ be a measurable function with support in the ball $B_R = B(0, R)$ of a radius $R < \infty$, and let
\[ \mathcal{K}_\epsilon(x) = \frac{1}{\epsilon^n} \mathcal{K}\left(\frac{x}{\epsilon}\right). \]
We consider the family of operators
\[ K_{\epsilon} f = \int_{\Omega} K_{\epsilon}(x - y) f(y) dy, \]  
(9)

\( \Omega \) being a bounded domain in \( R^n \).

For the given domain \( \Omega \) we define the larger domain
\[ \Omega_R = \{ x : \text{dist}(x, \Omega) \leq R \} \supseteq \Omega. \]

Let \( p(x) \) be a function defined in \( \Omega_R \) such that
\[ 1 \leq p(x) \leq P < \infty, \quad x \in \Omega_R. \]

Let also \( \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1 \) and
\[ Q = \left\{ \begin{array}{ll}
\sup_{x \in \Omega_R} q(x) = \frac{p_0}{p_0 - 1}, & \text{if } |E_1(p)| = 0 \\
\infty, & \text{if } |E_1(p)| > 0
\end{array} \right. \]
(11)

where \( E_1(p) = \{ x \in \Omega_R : p(x) = 1 \} \).

**Theorem 1.** Let \( K(x) \in L^Q(B_R) \) and let \( p(x) \) satisfy (10) and (8) for all \( x \) and \( y \in \Omega_R \). Then the operators \( K_{\epsilon} \) are uniformly bounded from \( L^{p(x)}(\Omega) \) into \( L^{p(x)}(\Omega_R) \):
\[ \|K_{\epsilon} f\|_{L^{p(x)}(\Omega)} \leq c \|f\|_{L^{p(x)}(\Omega)} \]
(12)

where \( c \) does not depend on \( \epsilon \).

**Theorem 2.** Let \( p(x) \) and \( K(x) \) satisfy the assumptions of Theorem 1 and
\[ \int_{B_R} K(y) dy = 1. \]
(13)

Then (9) is an identity approximation in \( L^{p(x)}(\Omega) \):
\[ \lim_{\epsilon \to 0} \|K_{\epsilon} f - f\|_{L^{p(x)}(\Omega_R)} = 0, \quad f(x) \in L^{p(x)}(\Omega). \]
(14)

Let
\[ f_\epsilon(x) = \frac{1}{\epsilon^n \|B(0,1)\|} \int_{y \in \Omega, |y-x| < \epsilon} f(y) dy \]
(15)

be the Steklov mean of the function \( f(y) \).

**Corollary 1.** Under the assumptions of Theorem 1 on \( p(x) \),
\[ \lim_{\epsilon \to 0} \|f_\epsilon - f\|_{L^{p(x)}(\Omega)} = 0. \]
(16)

**Remark 1.** The statement (16) is an analogue of mean continuity property for \( L^{p(x)} \)-spaces, but with respect to the averaged "shift" operator (15). In the standard form, the mean continuity property \( \lim_{h \to 0} \|f(x + h) - f(x)\|_p = 0 \), generally speaking, is not valid for variable exponents \( p(x) \) and, moreover, there exist functions \( p(x) \) and \( f(x) \in L^{p(x)}(\Omega) \) such that \( f(x + h_k) \notin L^{p(x)}(\Omega) \) for some \( h_k \to 0 \), see [2], Example 2.9 and Theorem 2.10.
Corollary 2. Let \( 1 \leq p(x) \leq P < \infty, x \in \mathbb{R}^n \), and \( p(x) \) satisfy the condition (8) in any ball in \( \mathbb{R}^n \) (where \( A \) may depend on the ball). Then \( C_0^\infty \) is dense in \( L^{p(x)}(\mathbb{R}^n) \).

Remark 2. As it was shown in [2], \( C_0^\infty(\Omega) \) is dense in \( L^{p(x)}(\Omega) \), \( 1 \leq p(x) \leq P < \infty \), without requiring that \( p(x) \) satisfies the condition (8).

Let \( \mathcal{W}^{m,p}(x) = \mathcal{W}^{m,p}(x)(\mathbb{R}^n) \) be the Sobolev type space of functions \( f(x) \in L^{p(x)}(\mathbb{R}^n) \) which have all the distributional derivatives \( D^j f(x) \in L^{p(x)}(\mathbb{R}^n), 0 \leq |j| \leq m \), and let

\[
\|f\|_{\mathcal{W}^{m,p}(x)} = \sum_{|j| \leq m} \|D^j f\|_{p(x)} .
\]

Theorem 3. Let \( p(x) \) satisfy the assumptions of Theorem 3. Then \( C_0^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{W}^{m,p}(x)(\mathbb{R}^n) \).

4. Proof of Theorem 1.

We assume that

\[
\|f\|_p \leq 1 .
\]

By (4)-(5) it suffices to show that

\[
I_p(K, f) = \int_{\Omega_R} |K, f(x)|^{p(x)} dx \leq c
\]

with \( c > 0 \) not depending on \( \epsilon \). By the Hölder inequality (3) it is easy to show that \( |K, f(x)| \leq c \) for all \( x \in \Omega_R \) and \( \epsilon \geq \epsilon^0(c = c(\epsilon^0) \) in this case). Therefore, it suffices to prove (18) for \( 0 < \epsilon \leq \epsilon^0 \) under some choice of \( \epsilon^0 \).

Let

\[
\Omega_R = \bigcup_{k=1}^N \Omega^k_R
\]

be any partition of \( \Omega_R \) into small parts \( \omega^k_R \) comparable with the given \( \epsilon : \)

\[
diam \omega^k_R \leq \epsilon, k = 1, 2, \ldots, N ; \quad N = N(\epsilon).
\]

We represent the integral in (18) as

\[
I_p(K, f) = \sum_{k=1}^N \int_{\omega^k_R} \left| \int_{\Omega^k_R} \mathcal{K}_\epsilon(x - y)f(y) dy \right|^{p(x) - p_k + p_k} dx
\]

with

\[
p_k = \inf_{x \in \Omega^k_R} p(x) \leq \inf_{x \in \omega^k_R} p(x)
\]

where some larger portions \( \Omega^k_R \supset \omega^k_R \) will be chosen later comparable with \( \epsilon : \)

\[
diam \Omega^k_R \leq m\epsilon, \quad m > 1 .
\]

We shall prove the uniform estimate

\[
A_k(x, \epsilon) = \left| \int_{\Omega} \mathcal{K}_\epsilon(x - y)f(y) dy \right|^{p(x) - p_k} \leq c, \quad x \in \omega^k_R
\]
where $c > 0$ does not depend on $x \in \omega^k_{R}$, $k$ and $\epsilon \in (0, \epsilon^o)$ with some $\epsilon^o > 0$. To this end, we first obtain the estimate

$$A_k(x, \epsilon) \leq c_1 \epsilon^{-n[p(x) - p_k]} , \quad x \in \Omega_R. \quad (23)$$

To get (23), we differ the cases $Q = \infty$ and $Q < \infty$.

Let $Q = \infty$. We have

$$A_k(x, \epsilon) \leq \left( \frac{M}{\epsilon^n} \int_{\Omega} \chi_{B(0, \epsilon R)}(y)|f(y)|dy \right)^{p(x) - p_k}$$

where $M = \sup_{B_R} |K(x)|$. By the Hölder inequality (3) and the assumption (17) we obtain

$$A_k(x, \epsilon) \leq \left( \frac{Mk}{\epsilon^n} \|\chi_{B(0, \epsilon R)}\|_q \right)^{p(x) - p_k} . \quad (24)$$

According to (2) we have

$$\|\chi_{B(0, \epsilon R)}\|_q = \sup_{E_{\infty}(q)} \chi_{B(0, \epsilon R)}(x) + \|\chi_{B(0, \epsilon R)}\|_q = 1 + \|\chi_{B(0, \epsilon R)}\|_q.$$  

In view of (6) we get

$$\|\chi_{B(0, \epsilon R)}\|_q \leq 1 + (\epsilon^n|B(0, R)|)^{\frac{1}{\theta}} \leq 2$$

under the assumption that

$$0 < \epsilon \leq |B(0, R)|^{-\frac{1}{\theta}} : = \epsilon^o . \quad (25)$$

Then (24) provides the estimate (23) with $c_1 = (2kM)^{P - p_0}$ if $2kM \geq 1$ and $c_1 = 1$ otherwise.

Let $Q < \infty$. The estimate (23) is obtained in a similar way. Indeed, applying the Hölder inequality (3) again, we arrive at

$$A_k(x, \epsilon) \leq \left( k\|K_\epsilon(x - y)\|_q \right)^{p(x) - p_k}.$$  

By (4)-(5) we have

$$\|K_\epsilon(x - y)\|_q = \frac{1}{\epsilon^n} \|K\left(\frac{x - y}{\epsilon}\right)\|_q \leq \frac{1}{\epsilon^n} \left( \int_{\Omega \setminus E_{\infty}(q)} \|K\left(\frac{x - y}{\epsilon}\right)\|_q^{q(y)} dy \right)^{\theta}$$

where $\theta = \frac{1}{Q}$ or $\theta = \frac{1}{q_0}$ depending on the fact whether the last integral in the parentheses is less or greater than 1, respectively. Hence,

$$\|K_\epsilon(x - y)\|_q \leq \frac{1}{\epsilon^n} \left( \int_{|y| < R, x - \epsilon y \in \Omega \setminus E_{\infty}(q)} |K(y)|^{q(x - \epsilon y)} dy \right)^{\theta} \leq \frac{1}{\epsilon^n} \left( |B_R| + \|K\|_Q^\theta \right)^{\theta} \leq c_2 \epsilon^{-n} \quad (27)$$

where $c_2 = \max\{c_3, c_3^\theta\}$, $c_3 = |B_R| + \|K\|_Q^\theta$. 

5
Therefore, from (26) and (27) we obtain (23) in the case $Q < \infty$ as well, with $c_1 = (c_2 k)^{p_p}$ if $c_2 > 1$ and $c_1 = 1$ otherwise.

The estimate (23) having been proved, we observe now that by (8)

$$p(x) - p_k = |p(x) - p(\xi_k)| \leq \frac{A}{\log \frac{1}{|x - \xi_k|}}$$

where $x \in \omega_R^k$, $\xi_k \in \Omega_R^k$. Evidently,

$$|x - \xi_k| \leq \text{diam} \Omega_R^k \leq m\epsilon$$

by (21). Therefore,

$$p(x) - p_k \leq \frac{A}{\log \frac{1}{m\epsilon}}$$

(28)

under the assumption that

$$0 < \epsilon \leq \frac{1}{2m} =: \epsilon^0.$$  

(29)

Then from (23) and (28)

$$A_k(x, \epsilon) \leq c_1 \epsilon^{-\frac{A}{\log \frac{1}{m\epsilon}}}, \quad x \in \omega_R^k,$$

(30)

c_1 \text{ not depending on } x \text{ and being given above. Then from (30)}

$$A_k(x, \epsilon) \leq c_4 := c_1 e^{2A}$$

for $x \in \omega_R^k$ and

$$0 < \epsilon \leq \epsilon^0_3 := \frac{1}{m^2}.$$  

(31)

Therefore, we have the uniform estimate (22) with $c = c_1 e^{2A}$ and $0 < \epsilon \leq \epsilon^0$, $\epsilon^0 = \min_{1 \leq k \leq 3} \epsilon^0_k$, $\epsilon^0_k$ being given by (25), (29) and (31).

Using the estimate (22) we obtain from (19)

$$I_p(K_\epsilon f) \leq c \sum_{k=1}^N \int_{\omega_R^k} \left| \int_{K_\epsilon(x-y)f(y)dy}^{p_k} \right| dx.$$  

Here $p_k$ are constants so that we may apply the usual Minkowsky inequality for integrals and obtain

$$I_p(K_\epsilon f) \leq c \sum_{k=1}^N \left\{ \int_{|y|<R} |K_\epsilon(y)| dy \left( \int_{\omega_R^k} |f(x-y)^{p_k} dx \right)^{\frac{1}{p_k}} \right\}^{p_k}$$

$$= c \sum_{k=1}^N \left\{ \int_{|y|<R} |K_\epsilon(y)| dy \left( \int_{x+ey \in \omega_R^k} |f(x)^{p_k} dx \right)^{\frac{1}{p_k}} \right\}^{p_k}.$$  

(32)
Obviously, the domain of integration in \( x \) in the last integral is embedded into the domain
\[
\bigcup_{y \in B_{k,R}} \{ x : x + y \in \omega_R^k \}
\] (33)
which already does not depend on \( y \). Now, we choose the sets \( \Omega^k_R \) in (20), which were not determined until now, as the sets (33). Then, evidently, \( \Omega^k_R \supset \omega_R^k \) and it is easily seen that
\[
diam \Omega^k_R \leq (1 + 2R) \varepsilon
\] (34)
so that the requirement (21) is satisfied with \( m = 1 + 2R \).

From (32) we have
\[
I_p(K \epsilon f) \leq c \sum_{k=1}^{N} \left\{ \int_{|y| < R} |K(y)| dy \right\}^{p_k} \int_{\Omega^k_R} |f(x)|^{p_k} dx
\]
\[
\leq c \left\{ \int_{|y| < R} |K(y)| dy \right\}^\theta \sum_{k=1}^{N} \int_{\Omega^k_R \cap \Omega} |f(x)|^{p_k} dx
\]
where \( \theta = P \) if \( \int_{|y| < R} |K(y)| dy \leq 1 \) and \( \theta = p_o \) otherwise. In view of (34), the covering \( \{ \omega_k = \Omega^k_R \cap \Omega \}_{k=1}^{N} \) has a finite multiplicity (that is, each point \( x \in \Omega \) belongs simultaneously not more than to a finite number \( n_o \) of the sets \( \omega_k \), \( n_o \leq 1 + (1 + 2R)^n \) in this case).

Therefore,
\[
I_p(K \epsilon f) \leq c_5 \int_{\Omega} |f(x)|^{\tilde{p}(x)} dx
\] (35)
where
\[
\tilde{p}(x) = \max_j p_j
\]
the maximum being taken with respect to all the sets \( \omega_j \) containing \( x \). Evidently, \( \tilde{p}(x) \leq p(x) \) for \( x \in \Omega \). Then from (35) and (4)-(5) we obtain the estimate
\[
I_p(K \epsilon f) \leq c_5 \|f\|_{\tilde{p}}^{\theta_1} \, \theta_1 < P
\]
with \( \theta_1 = \inf \tilde{p}(x) \) if \( \|f\|_{\tilde{p}} \leq 1 \) and \( \theta_1 = \sup \tilde{p}(x) \) otherwise. Applying the imbedding theorem (7), we arrive at the final estimate
\[
I_p(K \epsilon f) \leq c_6 \|f\|_p^{\theta_1} \leq c_6
\]

5. Proof of Theorem 2.

To prove (14), we use Theorem 1, which provides the uniform boundedness of the operators \( K \epsilon \) from \( L^{p(x)}(\Omega) \) into \( L^{p(x)}(\Omega_R) \). Then, by the Banach-Steinhaus theorem it suffices to verify that (14) holds for some dense set in \( L^{p(x)}(\Omega) \), for example, for step functions, according to property e) of the spaces \( L^{p(x)}(\Omega) \). So, it is sufficient to prove (14) for the characteristic function \( \chi_E(x) \) of any bounded measurable set \( E \subset \Omega \). We have
\[
K \epsilon (\chi_E) - \chi_E = \int_{B_{2R}} K(y) [\chi_E(x - \epsilon y) - \chi_E(x)] dy
\]
by (13). Hence
\[
\|K_\epsilon (\chi_E) - \chi_E\|_P \leq \int_{B_R} |\mathcal{K}(y)| \|\chi_E(\cdot - \epsilon y) - \chi_E(x)\|_P dy \to 0
\]
as \epsilon \to 0 by the Lebesgue dominated convergence theorem and the $P$-mean continuity of functions in $L^P$ with a constant $P$ ($P = \sup_{x \in \Omega_R} p(x)$ in this case). Then, by (7), also
\[
\|K_\epsilon (\chi_E) - \chi_E\|_P \to 0
\]
with $p = p(x) \leq P < \infty$. □

6. Proof of Corollaries
To obtain Corollary 1 from Theorem 1, it suffices to choose $\mathcal{K}(y) = \frac{1}{|B(0,1)|} \chi_{B(0,1)}(y)$.

Proof of Corollary 2. Let $\chi_N(x) = \chi_{B(0,N)}(x)$. Then the functions $f^N(x) = \chi_N(x)f(x)$ have compact support and approximate $f(x) \in L^p(x)(R^n)$:
\[
\|f - f^N\| \leq \|f - f^N\|_{L^p} = \left( \int_{|x| > N} |f(x)|^{p(x)} dx \right)^{\frac{1}{p}} \to 0
\]
as $N \to \infty$.

Therefore, we may consider $f(x)$ with a compact support in the ball $B_N$ from the very beginning. To approximate $f(x)$ by $C_0^\infty$, we use the identity approximation
\[
f_\epsilon(x) = \int_{R^n} \mathcal{K}_\epsilon (x - t)f(t) dt = \int_{|y| < 1} \mathcal{K}(y)f(x - \epsilon y) dy
\]
where $\mathcal{K}_\epsilon(x) = \frac{1}{\epsilon^n} \mathcal{K}\left(\frac{x}{\epsilon}\right)$ and $\mathcal{K}(y) \in C_0^\infty(R^n)$ with support in the ball $B_1$ and such that
\[
\int_{|y| < 1} \mathcal{K}(y) dy = 1.
\]
Then, evidently, $f_\epsilon(x) \in C_0^\infty(R^n)$ and has compact support because $f_\epsilon(x) \equiv 0$ if $|x| > N + \epsilon$.
Therefore, for $\epsilon < 1$,
\[
\|f_\epsilon - f\|_{L^p(R^n)} = \|K_\epsilon f - f\|_{L^p(B_{N+1})} \to 0
\]
as $\epsilon \to 0$.

Proof of Theorem 3.
The proof follows from Theorem 2 and Corollary 2 in two steps.
1°. Let $f(x) \in W^{m,p(x)}(R^n)$ and let $\mu(r), 0 \leq r \leq \infty$, be a smooth step-function: $\mu(r) \equiv 1$ for $0 \leq r \leq 1, \mu(r) \equiv 0$ for $r \geq 2, \mu(r) \in C_0^\infty(R^n)$ and $0 \leq \mu(r) \leq 1$. Then
\[
f^N(x) = \mu\left(\frac{|x|}{N}\right)f(x) \in W^{m,p(x)}(R^n)
\]
for every $N \in R_+^1$ and has compact support in $B_{2N}$. 

8
The functions (37) approximate \( f(x) \) in \( W^{m,p(x)}(\mathbb{R}^n) \). Indeed, denoting \( \nu_N(x) = 1 - \mu \left( \frac{|x|}{N} \right) \), so that \( \nu_N(x) \equiv 0 \) for \( |x| < N \), and using the Leibnitz formula for differentiation, we have

\[
\| f - f^N \|_{W^{m,p(x)}} = \sum_{|j| \leq m} \| D^j (\nu_N f) \|_p \leq \sum_{|j| \leq m} \sum_{0 \leq k \leq j} c_k \| D^k (\nu_N) D^{j-k} f \|_p
\]

\[
\leq \sum_{|j| \leq m} \| \nu_N D^j f \|_p + c \sum_{|j| \leq m} \sum_{0 \leq k \leq j} \| D^k (\nu_N) D^{j-k} f \|_p
\]

\[
\leq \sum_{|j| \leq m} \| \nu_N D^j f \|_p + c \sum_{|j| \leq m} \sum_{0 \leq k \leq j} \frac{1}{N^{|k|}} \| D^{j-k} f \|_p \to 0
\]

as \( N \to 0 \).

2. By the step 1° we may consider \( f(x) \in W^{m,p(x)} \) with compact support. Then we take \( K(y) \in C_0^\infty(\mathbb{R}^n) \) with support in the ball \( B_1 \) and such that \( \int_{|y| < 1} K(y) dy = 1 \) and arrange the approximation (36). Then, evidently, \( f_\epsilon \in C_0^\infty(\mathbb{R}^n) \). Indeed, for any \( j \) we have

\[
D^j f_\epsilon(x) = \frac{1}{\epsilon^{n+|j|}} \int_{|y| < 1} (D^j K) \left( \frac{x-t}{\epsilon} \right) f(t) dt \in C^\infty(\mathbb{R}^n)
\]

and \( f_\epsilon(x) \) has compact support because \( f_\epsilon(x) \equiv 0 \) if \( |x| > 1 + \lambda \), where \( \lambda = \sup_{x \in \text{supp } f} |x| \), \( \text{supp } f \) standing for support of \( f(x) \).

We have

\[
\| f_\epsilon(x) - f \|_{W^{m,p(x)}} \leq \sum_{|j| \leq m} \| D^j f - K_\epsilon(D^j f) \|_{L^p(x)(\mathbb{R}^n)}
\]

\[
= \sum_{|j| \leq m} \| D^j f - K_\epsilon(D^j f) \|_{L^p(x)(\Omega_1)}
\]

where \( \Omega_1 = \{ x : \text{dist}(x, \Omega) \leq 1 \} \), \( \Omega = \text{supp } f(x) \). It suffices to apply Theorem 2.

References


