MAXIMAL AND POTENTIAL OPERATORS IN VARIABLE EXponent MORREY SPACES

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Abstract. We prove the boundedness of the Hardy–Littlewood maximal operator on variable Morrey spaces \(L^{p(\cdot),\lambda(\cdot)}(\Omega)\) over a bounded open set \(\Omega \subset \mathbb{R}^n\) and a Sobolev type \(L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}\)-theorem for potential operators \(I^{\alpha(\cdot)}\), also of variable order. In the case of constant \(\alpha\), the limiting case is also studied when the potential operator \(I^{\alpha}\) acts into BMO space.

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1. Introduction

We introduce the variable exponent Morrey spaces \(L^{p(\cdot),\lambda(\cdot)}(\Omega)\) over an open set \(\Omega \subset \mathbb{R}^n\), well known in the case where \(p\) and \(\lambda\) are constant, see for instance [10], [17]. Last decade there was a real boom in the investigation of variable exponent Lebesgue spaces \(L^{p(\cdot)}(\Omega)\) and the corresponding Sobolev spaces \(W^{m,p(\cdot)}(\Omega)\). We refer to surveys [7], [12], [21] on the progress in this field, including topics of Harmonic Analysis and Operator Theory.

In this paper, within the framework of variable Morrey spaces \(L^{p(\cdot),\lambda(\cdot)}(\Omega)\) over bounded sets \(\Omega \subset \mathbb{R}^n\), we consider the Hardy–Littlewood maximal operator

\[Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy\]

and potential type operators

\[I^{\alpha(\cdot)}f(x) = \int_{\Omega} \frac{f(y) dy}{|x-y|^{n-\alpha(x)}}\]

of variable order \(\alpha(x)\).

We prove the boundedness of the maximal operator in Morrey spaces under the log-condition on \(p(\cdot)\) and \(\lambda(\cdot)\), in the case where \(\lambda(x) \equiv 0\) this result being due to L. Diening [5]. For potential operators, under the same log-conditions and the assumptions \(\inf_{x \in \Omega} \alpha(x) > 0\), \(\sup_{x \in \Omega} [\lambda(x) + \alpha(x)p(x)] < n\), we prove a Sobolev type \(L^{p(\cdot),\lambda(\cdot)} \to L^{q(\cdot),\lambda(\cdot)}\)-theorem. In the case \(\lambda(x) \equiv 0\) the Sobolev Theorem for variable Lebesgue spaces is known to be obtained – via the Hedberg approach – from the boundedness of the maximal operator, see [20].
As a corollary of the Sobolev Theorem, we derive the corresponding boundedness of the fractional maximal operator

\[ M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x, r)|^{1-\frac{\alpha(x)}{n}}} \int_{B(x, r)} |f(y)|dy. \]

In the case of constant \( \alpha \), we also prove a boundedness theorem in the limiting case \( p(x) = \frac{n-\lambda(x)}{\alpha(x)} \), when the potential operator \( I_\alpha \) acts from \( L^{p(\cdot),\lambda(\cdot)} \) into \( \text{BMO} \).

**Notation:**
- \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space;
- \( \Omega \) is a non-empty open set in \( \mathbb{R}^n \);
- \( d_\Omega \) denotes the diameter of \( \Omega \);
- \( \chi_E \) is a characteristic function of a measurable set \( E \subset \mathbb{R}^n \);
- \( |E| \) is the Lebesgue measure of \( E \);
- \( B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \} \), \( \tilde{B}(x, r) = B(x, r) \cap \Omega \);
- by \( c \) and \( C \) we denote various absolute positive constants, which may have different values even in the same line.

2. **Preliminaries on Variable Exponent Lebesgue Spaces**

Let \( p(\cdot) \) be a measurable function on \( \Omega \) with values in \([1, \infty)\). We assume that

\[ 1 \leq p_- \leq p(x) \leq p_+ < \infty, \]

where we use the standard notation

\[ p_- := \text{ess inf}_{x \in \Omega} p(x) \quad \text{and} \quad p_+ := \text{ess sup}_{x \in \Omega} p(x). \]

By \( L^{p(\cdot)}(\Omega) \) we denote the space of all measurable functions \( f \) on \( \Omega \) such that

\[ I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)}dx < \infty. \]

Equipped with the norm

\[ \| f \|_{p(\cdot)} = \inf \left\{ \eta > 0 : I_{p(\cdot)} \left( \frac{f}{\eta} \right) \leq 1 \right\}, \]

this is a Banach function space. As is known, the following inequalities hold

\[ \| f \|_{p(\cdot)}^{p_+} \leq I_{p(\cdot)}(f) \leq \| f \|_{p(\cdot)}^{p_-} \quad \text{if} \quad \| f \|_{p(\cdot)} \leq 1, \]

\[ \| f \|_{p(\cdot)}^{p_-} \leq I_{p(\cdot)}(f) \leq \| f \|_{p(\cdot)}^{p_+} \quad \text{if} \quad \| f \|_{p(\cdot)} \geq 1, \]

from which it follows that

\[ c_1 \leq \| f \|_{p(\cdot)} \leq c_2 \quad \implies \quad c_3 \leq I_{p(\cdot)}(f) \leq c_4 \]

and

\[ C_1 \leq I_{p(\cdot)}(f) \leq C_2 \quad \implies \quad C_3 \leq \| f \|_{p(\cdot)} \leq C_4, \]
with $c_3 = \min (c_1^{p_+}, c_1^{p_-})$, $c_4 = \max (c_2^{p_+}, c_2^{p_-})$, $C_3 = \min \left( C_1^{1/p_-}, C_1^{1/p_+} \right)$ and $C_4 = \max \left( C_2^{1/p_-}, C_2^{1/p_+} \right)$.

As usual, we denote by $p'(\cdot)$ the conjugate exponent given by $p'(x) = \frac{p(x)}{p(x) - 1}$, $x \in \Omega$. The Hölder inequality is valid in the form
\[
\int_{\Omega} |f(x)g(x)| \, dx \leq \left( \frac{1}{p_-} + \frac{1}{p'_-} \right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \tag{6}
\]

If $\Omega$ is bounded and $p(x) \leq q(x)$, there holds the embedding
\[
L^q(\cdot)(\Omega) \hookrightarrow L^p(\cdot)(\Omega). \tag{7}
\]

For the basics of variable exponent Lebesgue spaces we refer to [22], [16].

The $L^p(\cdot)(\cdot)$-boundedness of the Hardy–Littlewood maximal operator was proved by L. Diening [5] under the conditions
\[
1 < p_- \leq p(x) \leq p_+ < \infty \tag{8}
\]
and
\[
|p(x) - p(y)| \leq \frac{A}{\ln |x - y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega, \tag{9}
\]
where $A > 0$ does not depend on $x, y$.

The proof of the boundedness of the maximal operator was based on the following pointwise estimate.

**Lemma 1.** ([5]) Let $\Omega$ be bounded and let $p(\cdot)$ satisfy conditions (8), (9). Then there exists a constant $C > 0$ such that, for all $\|f\|_{p(\cdot)} \leq 1$,
\[
(Mf(x))^{\frac{p(x)}{p_-}} \leq C \left[ M \left( |f(\cdot)|^{\frac{p(\cdot)}{p_-}} \right)(x) + 1 \right]. \tag{10}
\]

We will also make use of the following statement proved in [20], Theorem 1.17.

**Theorem 1.** Let $p(\cdot)$ satisfy assumptions (1) and (9) and $\beta(\cdot)$ satisfy the conditions
\[
\sup_{x \in \Omega} \beta(x) < \infty, \quad \inf_{x \in \Omega} \beta(x)p(x) > n. \tag{11}
\]

Then the estimate
\[
\left\| \chi_{\mathbb{R}^n \setminus B(x,r)}^{(\cdot)} \right\|_{p(\cdot)} \leq C r^{n-\beta(x)} \tag{12}
\]
is valid, where the constant $C > 0$ depends on $\sup_{x \in \Omega} \beta(x)$ and $\inf_{x \in \Omega} [\beta(x)p(x) - n]$, but does not depend on $x$ and $r$.

We note that the logarithmic condition (9) is usually called the log-Hölder continuity or the Dini–Lipschitz condition.
3. Variable Exponent Morrey Spaces

3.1. Definition. Let $\lambda(\cdot)$ be a measurable function on $\Omega$ with values in $[0, n]$. We define the variable Morrey space $L^{p(\cdot), \lambda(\cdot)}(\Omega)$ as the set of all integrable functions $f$ on $\Omega$ such that

$$I_{p(\cdot), \lambda(\cdot)}(f) := \sup_{x \in \Omega, r > 0} r^{-\lambda(x)} \int_{B(x, r)} |f(y)|^{p(y)} dy < \infty. \quad (13)$$

The norm in the space $L^{p(\cdot), \lambda(\cdot)}(\Omega)$ may be introduced in two forms,

$$\|f\|_1 = \inf \left\{ \eta > 0 : I_{p(\cdot), \lambda(\cdot)} \left( \frac{f}{\eta} \right) \leq 1 \right\}$$

and

$$\|f\|_2 = \sup_{x \in \Omega, r > 0} \left\| r^{-\frac{\lambda(x)}{p(x)}} f \chi_{B(x, r)} \right\|_{p(\cdot)},$$

which actually coincide, as shown in Lemma 3. First we need the following lemma.

Lemma 2. For every $f \in L^{p(\cdot), \lambda(\cdot)}(\Omega)$, the inequalities

$$\|f\|_i^{p_i^+} \leq I_{p(\cdot), \lambda(\cdot)}(f) \leq \|f\|_i^{p_i^-} \quad \text{if} \quad \|f\|_i \leq 1, \quad (14)$$

$$\|f\|_i^{p_i^-} \leq I_{p(\cdot), \lambda(\cdot)}(f) \leq \|f\|_i^{p_i^+} \quad \text{if} \quad \|f\|_i \geq 1 \quad (15)$$

are valid, $i = 1, 2$.

Proof. Let

$$F(x, r; \eta) = \frac{1}{r^{\lambda(x)}} \int_{B(x, r)} \left| \frac{f(y)}{\eta} \right|^{p(y)} dy. \quad (16)$$

For every $(x, r) \in \Omega \times (0, d_{\Omega})$, the function $F(x, r; \eta)$ is decreasing in $\eta \in (0, \infty)$. We have

$$\sup_{x \in \Omega, r > 0} F(x, r; 1) = I_{p(\cdot), \lambda(\cdot)}(f) \quad (17)$$

and by the definition of the norm $\| \cdot \|_1$,

$$\sup_{x \in \Omega, r > 0} F(x, r; \|f\|_1) = 1. \quad (18)$$

Then from (17)–(18), by the monotonicity of $F(x, r; \eta)$ in $\eta$, inequalities (14)–(15) with $i = 1$ follow. To cover the case $i = 2$, that is, the case of the norm

$$\|f\|_2 = \sup_{x \in \Omega, r > 0} \|g_{x, r}\|_{p(\cdot)},$$

where $g_{x, r} = r^{-\frac{\lambda(x)}{p(x)}} f \chi_{B(x, r)}(\cdot)$, we make use of inequalities (2)–(3) of the $L^{p(\cdot)}$-norm and have

$$\|g_{x, r}\|_{p(\cdot)}^{p_i^+} \leq I_{p(\cdot)}(g_{x, r}) \leq \|g_{x, r}\|_{p(\cdot)}^{p_i^-} \quad \text{if} \quad \|g_{x, r}\|_{p(\cdot)} \leq 1,$$

and similarly for the case $\|g_{x, r}\|_{p(\cdot)} \geq 1$. Taking the supremum with respect to $x$ and $r$, we obtain (14)–(15) for $i = 2$. \qed
Lemma 3. For every \( f \in L^{p(\cdot, \lambda(\cdot))}(\Omega) \) we have

\[
\|f\|_2 = \|f\|_1.
\]

Proof. We note that

\[
\|f\|_2 = \sup_{x \in \Omega, r > 0} \left\{ \mu_{x,r} > 0 : F(x, r, \mu_{x,r}) = 1 \right\},
\]

where \( F(x, r; \eta) \) is function (16). From the equality \( F(x, r; \mu_{x,r}) = 1 \) and the inequality \( F(x, r; \|f\|_1) \leq 1 \) following from (18), by the monotonicity of the function \( F(x, r; \eta) \) with respect to \( \eta \), we conclude that

\[
\|f\|_2 \leq \|f\|_1.
\]

From relations (14) we easily derive

\[
\|f\|_1 \leq \begin{cases} 
\|f\|_2^{\frac{p-1}{p}} & \text{if } \|f\|_1 \leq 1, \\
\|f\|_2 & \text{if } \|f\|_1 \geq 1, \|f\|_2 \leq 1, \\
\|f\|_2^{\frac{p-1}{p}} & \text{if } \|f\|_1 \geq 1, \|f\|_2 \geq 1.
\end{cases}
\]

Substituting here \( \frac{f}{\|f\|_2} \) instead of \( f \), we obtain

\[
\left\| \frac{f}{\|f\|_2} \right\|_1 \leq 1,
\]

that is, \( \|f\|_1 \leq \|f\|_2 \), which completes the proof. \( \square \)

By the coincidence of the norms we put

\[
\|f\|_{p(\cdot), \lambda(\cdot)} := \|f\|_1 = \|f\|_2.
\]

Remark 1. When the open set \( \Omega \) is bounded, the supremum defining the norm \( \| \cdot \|_2 \) is always reached for values of \( r \) less than \( d_\Omega \). Indeed, if \( r \geq d_\Omega \) we have

\[
\left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{\bar{B}(x,r)} \right\|_{p(\cdot)} \leq \left\| d_\Omega^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{\bar{B}(x,d_\Omega)} \right\|_{p(\cdot)}.
\]

Lemma 5 below provides another equivalent norm on \( L^{p(\cdot, \lambda(\cdot))}(\Omega) \) when \( |\Omega| < \infty \). Basically, it states that in case \( \lambda(\cdot) \) is log-continuous, there is no difference in taking the parameter \( \lambda \) depending on \( x \) or \( y \). Lemma 5 is an immediate consequence of the following simple lemma.

Lemma 4. Let \( \Omega \) be a bounded open set and \( \lambda(\cdot) \) satisfy the logarithmic condition

\[
|\lambda(x) - \lambda(y)| \leq \frac{A_\lambda}{-\ln |x-y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega.
\]

Then

\[
\frac{1}{C}r^{-\lambda(y)} \leq r^{-\lambda(x)} \leq Cr^{-\lambda(y)}
\]

for all \( x, y \in \Omega \) such that \( |x - y| \leq r \), with the constant \( C = e^{A_\lambda} \) not depending on \( x, y \) and \( r \).
Proof. Since the set $\Omega$ is bounded and $\lambda(\cdot)$ is a bounded function, it suffices to check (20) for small $r \leq 1$. It is easy to see that (20) is equivalent to

$$|\lambda(x) - \lambda(y)| \ln \frac{1}{r} \leq C_1 : = \ln C = A_\lambda,$$

which is valid since $\ln \frac{1}{r} \leq \ln \frac{1}{|x-y|}$.

Lemma 5. If $\Omega$ is bounded and $\lambda(\cdot)$ is log-Hölder continuous, then the functional

$$\|f\|_3 := \sup_{x \in \Omega, r > 0} \left\| r^{-\frac{\lambda(\cdot)}{p(\cdot)}} f \chi_{B(x,r)} \right\|_{p(\cdot)}$$

(22)

defines an equivalent norm in $L^{p(\cdot), \lambda(\cdot)}(\Omega)$.

The definitions above recover the classical Morrey spaces (see, for example, [17], Ch. 4), that is, $L^{p(\cdot), \lambda(\cdot)}(\Omega) = L^{p, \lambda}(\Omega)$ if $p(\cdot) \equiv p$ and $\lambda(\cdot) \equiv \lambda$ are constant. Furthermore, if $\lambda(x) \equiv 0$, then $L^{p(\cdot), \lambda(\cdot)}(\Omega)$ coincides with the Lebesgue space $L^{p(\cdot)}(\Omega)$.

3.2. Embeddings of variable Morrey spaces. In Lemma 7 we prove embeddings of the Morrey spaces with variable $p(\cdot)$ and $\lambda(\cdot)$, known for constant exponents (see [17], Theorem 4.3.6 or [10], Ch. III, Proposition 1.1). To this end, we first need the estimate given in the following lemma which was obtained in [11] in the framework of general metric measure spaces setting; we give its another proof for the sake of completeness. This estimate serves better for our goals than the known ([6]) estimate

$$\|\chi_{B(x,r)}\|_{p(\cdot)} \leq C_{1} \left[ \mu B(x,r) \right]^{\frac{1}{p(x)}},$$

(23)

with $C > 0$ not depending on $x \in \Omega$ and $r > 0$ (for small $r$ one may take $C = e^{A_{\delta}}$, where $A > 0$ is the constant from the log-condition).

Lemma 6. Let $\Omega$ be a bounded open set in a metric measure space $(X, d, \mu)$ where the measure $\mu$ satisfies the lower Ahlfors condition $\mu B(x,r) \geq c r^\delta$, $\delta > 0$, and let $p(\cdot)$ satisfy the log-condition on $\Omega$. Then

$$\left\| \chi_{\tilde{B}(x,r)} \right\|_{p(\cdot)} \leq C \left[ \mu B(x,r) \right]^{\frac{1}{p(x)}}$$

(23)

with $C > 0$ not depending on $x \in \Omega$ and $r > 0$ (for small $r$ one may take $C = e^{A_{\delta}}$, where $A > 0$ is the constant from the log-condition).

Proof. Let $x \in \Omega$ and $0 < r < d_{Q}$. Since $p(\cdot)$ is log-Hölder continuous and the lower Ahlfors condition holds, it is easy to check that

$$\frac{1}{C} \mu B(x,r) \leq \left[ \mu B(x,r) \right]^{\frac{p(y)}{p(x)}} \leq C \mu B(x,r)$$

(24)

for all $y \in \tilde{B}(x,r)$, where $C \geq 1$ does not depend on $x$, $y$ and $r$. Hence for $C_1 = C^{\frac{1}{p(y)}}$ we have

$$\int_{\tilde{B}(x,r)} \frac{d\mu(y)}{C_1^{p(y)} \left[ \mu B(x,r) \right]^{\frac{p(y)}{p(x)}}} \leq \int_{\tilde{B}(x,r)} \frac{d\mu(y)}{\mu B(x,r)} \leq 1.$$
Then
\[ \left\| \chi_{\overline{B}(x,r)} \right\|_{p(\cdot)} = \inf \left\{ \eta > 0 : \int_{\overline{B}(x,r)} \eta^{-p(y)} \, d\mu(y) \leq 1 \right\} \leq C_1 \left[ \mu B(x,r) \right]^{\frac{1}{q(y)}}. \]

**Lemma 7.** Let \( \Omega \) be bounded, \( 0 \leq \lambda(x) \leq n \) and \( 0 \leq \mu(x) \leq n \). If \( p(\cdot) \) and \( q(\cdot) \) are log-Hölder continuous, \( p(x) \leq q(x) \) and
\[
\frac{n - \lambda(x)}{p(x)} \geq \frac{n - \mu(x)}{q(x)},
\]
then
\[
L^{q(\cdot),\mu(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(\Omega).
\]

**Proof.** Let \( \|f\|_{q(\cdot),\mu(\cdot)} \leq 1 \). This is equivalent to assuming that \( I_{q(\cdot),\mu(\cdot)}(f) \leq 1 \) (see Lemma 2). We have to show that \( I_{p(\cdot),\lambda(\cdot)}(f) \leq C \) for some \( C > 0 \) not depending on \( f \). Let \( x \in \Omega \) and \( r \in (0, d_\Omega) \). Applying the Hölder inequality (6) with the exponent \( p_1(x) = \frac{q(x)}{p(x)} \), we get
\[
r^{-\lambda(x)} \int_{\overline{B}(x,r)} |f(y)|^{p(y)} \, dy \leq 2 r^{-\lambda(x)} \left\| f^{p(\cdot)} \chi_{\overline{B}(x,r)} \right\|_{p_1(\cdot)} \left\| \chi_{\overline{B}(x,r)} \right\|_{q_1(\cdot)}.
\]
By Lemma 6 we have
\[
\left\| \chi_{\overline{B}(x,r)} \right\|_{q_1(\cdot)} \leq C r^{n \left(1 - \frac{p(x)}{q(x)}\right)}.
\]
For the norm \( \left\| f^{p(\cdot)} \chi_{\overline{B}(x,r)} \right\|_{p_1(\cdot)} \) we have the estimate
\[
\left\| f^{p(\cdot)} \chi_{\overline{B}(x,r)} \right\|_{p_1(\cdot)} = \inf \left\{ \eta > 0 : \int_{\overline{B}(x,r)} |f(y)|^{q(y)} \eta^{-\frac{q(y)}{p(y)}} \, dy \leq 1 \right\}
\]
\[
\leq A^{p_1(x)} r^{\mu(x) \frac{p_1(x)}{q_1(x)}},
\]
where \( A \) is the constant from the inequality
\[
\frac{1}{A} r^{\frac{\mu(x)}{q_1(y)}} \leq r^{\frac{\mu(x)p(x)}{q(x)p(y)}} \leq A r^{\frac{\mu(x)}{q_1(y)}}
\]
\((A \geq 1 \text{ not depending on } x, y, r)\). Indeed, by (30),
\[
\int_{\overline{B}(x,r)} \left( |f(y)| \left[ A^{p_1(x)} r^{\mu(x) \frac{p_1(x)}{q_1(x)}} \right]^{-\frac{q(y)}{p(y)}} \right) \, dy \leq \int_{\overline{B}(x,r)} \left( A^{-1} |f(y)| \left[ r^{-\mu(x) \frac{p(x)}{q(x)p(y)}} \right]^{q(y)} \right) \, dy
\]
\[
\leq r^{-\mu(x)} \int_{\overline{B}(x,r)} |f(y)|^{q(y)} \, dy \leq 1.
\]
which proves (29). Making use of estimates (28) and (29) in (27), we get
\[
\int_{B(x,r)} |f(y)|^{p(y)} dy \leq C r^{n - \lambda(x) - \frac{p(x)}{q(x)}[\mu(x)]},
\]
which is dominated by \( r^\mu(x) \) under condition (25). Then \( I_{p_\cdot,\lambda_\cdot}(f) \leq c \), with \( c \) not depending on \( x \) and \( r \). Therefore \( \|f\|_{p_\cdot,\lambda_\cdot} \leq C \) (see Lemma 2), which proves embedding (26).

To complete the proof, it remains to note that estimate (30) is a consequence of the log-Hölder continuity of \( \frac{p(x)}{q(x)} \).

\[ \square \]

4. The Maximal Operator in Variable Exponent Morrey Spaces

Following the notation above we put \( \lambda_+ := \operatorname{ess} \sup_{x \in \Omega} \lambda(x) \). In the sequel we suppose that
\[
0 \leq \lambda(x) \leq \lambda_+ < n, \quad x \in \Omega.
\]
(31)

For the constant exponents \( p(x) \equiv p \) and \( \lambda(x) \equiv \lambda \) the following theorem was proved in [4].

**Theorem 2.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). Under conditions (31), (8) and (9), the maximal operator \( M \) is bounded in the space \( L^{p_\cdot,\lambda_\cdot}(\Omega) \).

**Proof.** We have to show that
\[
I_{p_\cdot,\lambda_\cdot}(Mf) \leq C \quad \text{for all } f \text{ with } \|f\|_{p_\cdot,\lambda_\cdot} \leq c,
\]
where \( c > 0 \) and \( C = C(c) \) does not depend on \( f \). We continue the function \( f \) by zero beyond the set \( \Omega \) whenever necessary, and obtain
\[
\int_{\tilde{B}(x,r)} (Mf(y))^{p(y)} dy = \int_{\mathbb{R}^n} \left( (Mf(y))^{\frac{p(y)}{p-}} \right)^{p-} \chi_{\tilde{B}(x,r)}(y) dy.
\]
Since \( \Omega \) is bounded, \( \|f\|_{p,\lambda} \geq c \|f\|_p \) with some \( c > 0 \) depending only on \( d_\Omega \) and \( \lambda_+ \). Then the pointwise estimate (10) is applicable, which yields
\[
\int_{\tilde{B}(x,r)} [Mf(y)]^{p(y)} dy \leq C \left( \int_{\Omega} \left( |f(y)|^{\frac{p(y)}{p-}} \right)^{p-} \chi_{\tilde{B}(x,r)}(y) dy \right) + \int_{\Omega} \chi_{\tilde{B}(x,r)}(y) dy.
\]
By the Fefferman–Stein inequality (for constant \( p \in (1, \infty) \))
\[
\int_{\mathbb{R}^n} (Mg)(y)^p h(y) dy \leq \int_{\mathbb{R}^n} g(y)^p (Mh)(y) dy,
\]
valid for all non-negative functions \( g, h \) (see [9]), we get
\[
\int_{\tilde{B}(x,r)} (Mf(y))^{p(y)} dy \leq C \left( \int_{\Omega} |f(y)|^{p(y)} M\chi_{\tilde{B}(x,r)}(y) dy + r^n \right).
\]
The following estimate is known
\[ M \chi_{B(x,r)}(y) \leq \frac{4^n r^n}{(|x - y| + r)^n}, \quad x, y \in \mathbb{R}^n, \quad r > 0, \]
see [3], Lemma 2. Therefore
\[
\int_{\tilde{B}(x,r)} (Mf(y))^{p(y)} \, dy \\
\leq C \left( \int_{\tilde{B}(x,2r)} |f(y)|^{p(y)} \, dy + \sum_{j=1}^{\infty} \int_{\tilde{B}(x,2^{j+1}r) \setminus \tilde{B}(x,2^j r)} \frac{r^n |f(y)|^{p(y)}}{(|x - y| + r)^n} \, dy + r^n \right) \\
\leq C \left( r^\lambda(x) + \sum_{j=1}^{\infty} \frac{(2^{j+1}r)^\lambda(x)}{(2^j + 1)^n} + r^n \right) \leq C \left( r^\lambda(x) + r^n \right) \leq C \left( r^\lambda(x) \right),
\]
which proves the uniform estimate \( I_{p(\cdot),\lambda(\cdot)}(f) \leq C \) and completes the proof. □

Let \( M^2 \) be the sharp maximal function defined by
\[
M^2 f(x) := \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{\tilde{B}(x,r)} \left| f(y) - f_{\tilde{B}(x,r)} \right| \, dy,
\]
where
\[
f_{\tilde{B}(x,r)} = \frac{1}{|\tilde{B}(x,r)|} \int_{\tilde{B}(x,r)} f(z) \, dz
\]
is the mean value of \( f \) over \( \tilde{B}(x,r) \). From the boundedness of the maximal operator \( M \) and the pointwise inequality
\[
M^2 f(x) \leq 2Mf(x), \quad x \in \Omega,
\]
we can derive the following statement.

**Corollary 1.** Under the same conditions of Theorem 2, the sharp maximal operator \( M^2 \) is bounded in \( L^{p(\cdot),\lambda(\cdot)}(\Omega) \).

### 5. Potential Operators in Variable Morrey Spaces

Below we need to assume that \( \alpha(\cdot) \) also satisfies the log-condition
\[
|\alpha(x) - \alpha(y)| \leq \frac{C}{-\ln |x - y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega.
\]
(33)

The next theorem in the case of constant \( p \) and \( \alpha \) was proved in [1].

**Theorem 3.** Let \( \Omega \) be bounded. Under conditions (8), (9), (33) and the conditions
\[
\inf_{x \in \Omega} \alpha(x) > 0, \quad \sup_{x \in \Omega} [\lambda(x) + \alpha(x)p(x)] < n,
\]
(34)
the operator \( I_{\alpha(\cdot)} \) is bounded from \( L^{p(\cdot),\lambda(\cdot)}(\Omega) \) to \( L^{q(\cdot),\lambda(\cdot)}(\Omega) \), where \( \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n - \lambda(x)} \).
Proof. Let \( \|f\|_{p(\cdot),\lambda(\cdot)} \leq 1 \). As always, we continue the function \( f \) by zero beyond the set \( \Omega \). We use the standard decomposition

\[
I^{\alpha(\cdot)} f(x) = \left( \int_{B(x,2r)} + \int_{\mathbb{R}^n \setminus B(x,2r)} \right) f(y) |x-y|^\alpha(x) - n \, dy =: F(x,r) + G(x,r). \tag{35}
\]

The pointwise estimate

\[
|F(x,r)| \leq C r^\alpha(x) M f(x) \tag{36}
\]

with a constant \( C > 0 \) not depending on \( f \) and \( x \) is well known in the case of constant \( \alpha \) and is also valid for variable \( \alpha(\cdot) \), under the condition \( \alpha(x) > 0 \) (see [20], formula (56)). For \( G(x,r) \) we have

\[
|G(x,r)| \leq C \sum_{j=1}^{\infty} \int_{B(x,2^{j+1}r) \setminus B(x,2^j r)} |f(y)| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} |x-y|^\alpha(x) - n + \frac{\lambda(x)}{p(\cdot)} \, dy,
\]

where the series turns out to be a finite sum \( \sum_{j=1}^{N} \) for any fixed \( r > 0 \), with \( N = N(r) \) tending to infinity as \( r \to 0 \). Since the set \( \Omega \) is bounded and \( p(\cdot) \) satisfies the log-condition, we also have

\[
|G(x,r)| \leq C \sum_{j=1}^{\infty} \int_{B(x,2^{j+1}r) \setminus B(x,2^j r)} |f(y)| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} |x-y|^\alpha(x) - n + \frac{\lambda(x)}{p(\cdot)} \, dy. \tag{37}
\]

Applying the Hölder inequality, we get

\[
|G(x,r)| \leq C \sum_{j=1}^{\infty} \left\| |x-y|^\alpha(x) - n + \frac{\lambda(x)}{p(\cdot)} \right\|_{p^*(\cdot),\mathbb{R}^n \setminus B(x,2^j r)} \left\| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)}.
\]

(38)

The factor \( \left\| (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \right\|_{p(\cdot),B(x,2^{j+1}r)} \) is uniformly bounded. Indeed, to see this, in view of (4)–(5) it suffices to show the boundedness of the corresponding modular, i.e. that

\[
I_{p(\cdot)} \left( (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{(B(x,2^{j+1}r))} \right) \leq C < \infty,
\]

which is valid, since

\[
I_{p(\cdot)} \left( (2^j r)^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{(B(x,2^{j+1}r))} \right) \leq 2^{j+1} r^{-\lambda(x)} \int_{B(x,2^{j+1} r)} |f(y)|^{p(y)} \, dy \leq C < \infty
\]

by the definition in (13). Therefore, from (38), by Theorem 1 we obtain

\[
|G(x,r)| \leq C_1 \sum_{j=1}^{\infty} (2^j r)^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}} \leq C_2 r^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}}, \tag{39}
\]
where the series defining $C_2 = C_1 \sum_{j=1}^{\infty} 2^{-a_j}$, $a = \frac{1}{p^+} \inf_{x \in \Omega} \left[ n - \lambda(x) - \alpha(x)p(x) \right] > 0$, is convergent.

Thus, from (36) and (39) we have
\[
|I^{\alpha(x)} f(x)| \leq C r^{\alpha(x)} M f(x) + C_2 r^{\alpha(x) - \frac{n-\lambda(x)}{p(x)}}.
\]
Minimizing with respect to $r$, at $r = (Mf(x))^{-\frac{p(x)}{n-\lambda(x)}}$ we get
\[
|I^{\alpha(x)} f(x)| \leq c (Mf(x))^{\frac{p(x)}{n-\lambda(x)}}.
\]
Hence, by Theorem 2, we have
\[
\int_{B(x,r)} |I^{\alpha(x)} f(y)|^{q(y)} dy \leq c \int_{B(x,r)} (Mf(y))^{p(y)} dy \leq cr^{\lambda(x)},
\]
which completes the proof of the theorem. \[\square\]

The statement of the following corollary in the case of constant exponents $p, \lambda$ and $\alpha$ is known, see [8], Lemma 4, and [2], Corollary 4.4. Note that in the case of constant $p, \lambda$ and $\alpha$, the norm equivalence of $I^\alpha f$ and $M^\alpha f$ in Morrey spaces is also known, see [2].

**Corollary 2.** Under the assumptions of Theorem 3 the fractional maximal operator $M^{\alpha(x)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\mu(\cdot)}(\Omega)$.

**Proof.** The result follows from Theorem 3 in view of the pointwise estimate
\[
M^{\alpha(x)} f(x) \leq c I^{\alpha(x)}(|f|)(x), \quad 0 < \alpha(x) < n,
\]
where $c > 0$ does not depend on $f$ and $x$. This inequality, well known for constant $\alpha$, is also valid for variable $\alpha(x)$ with
\[
c = \sup_{x \in \Omega} \left( \frac{n}{|S^{n-1}|} \right)^{1-\frac{\alpha(x)}{n}} < \infty,
\]
where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. To prove (40), we observe that $I^{\alpha(x)}(|f|)(x) \geq \int_{B(x,r)} \frac{|f(y)| \, dy}{|x-y|^{n-\alpha(x)}}$ for any $x \in \Omega$ and $r > 0$. Since $|B(x,r)| = \frac{|S^{n-1}|}{n} r^n$, we have
\[
\frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}} \int_{B(x,r)} |f(y)| \, dy \leq \left( \frac{n}{|S^{n-1}|} \right)^{1-\frac{\alpha(x)}{n}} \int_{|x-y|^{\alpha(x)}} |f(y)| \, dy,
\]
whence (40) follows. \[\square\]

The following statement holds by Theorem 3 and embedding (26).

**Theorem 4.** Let the set $\Omega$ be bounded and $p(\cdot)$ satisfy conditions (8) and (9). Assume also that $\alpha(\cdot)$ and $\lambda(\cdot)$ are log-Hölder continuous and condition (34) is satisfied. Then the operator $I^{\alpha(x)}$ is bounded from $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^{q(\cdot),\mu(\cdot)}(\Omega)$,
where \( q(\cdot) \) is any exponent satisfying log-condition (9) and the condition \( 1 \leq q(x) \leq \frac{p(x)n-\lambda(x)}{n-\alpha(x)p(x)} \), and \( \mu(\cdot) \) is defined by the condition

\[
\frac{n-\mu(x)}{q(x)} = \frac{n-\lambda(x)}{p(x)} - \alpha(x).
\]

In particular, one may take

\[
\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \quad \text{and} \quad \mu(x) = \frac{n\lambda(x)}{n-\alpha(x)p(x)}.
\]

For constant exponents the statement of Theorem 4 in the case of (41) may be found, for example, in [4] (see Corollary on page 277).

Similarly to Corollary 2 we derive the following result.

**Corollary 3.** Under the assumptions of Theorem 4, the fractional maximal operator \( M^{\alpha(\cdot)} \) is bounded from \( L^{p(\cdot),\lambda(\cdot)}(\Omega) \) to \( L^{q(\cdot),\mu(\cdot)}(\Omega) \).

**6. Potential Operators: the Limiting Case**

In this section we study the limiting case in (34), that is, we consider the critical exponent

\[
p(x) = \frac{n-\lambda(x)}{\alpha(x)}.
\]

**Theorem 5.** Let \( \Omega \) be bounded, \( p(\cdot) \) satisfy conditions (8), (9) and \( \inf_{x \in \Omega} \alpha(x) > 0 \). In the case of exponent (42) the operator \( M^{\alpha(\cdot)} \) is bounded from \( L^{p(\cdot),\lambda(\cdot)}(\Omega) \) to \( L^{\infty}(\Omega) \):

\[
\|M^{\alpha(\cdot)}f\|_{\infty} \leq C \|f\|_{p(\cdot),\lambda(\cdot)}.
\]

**Proof.** Let \( x \in \Omega \) and \( r > 0 \). By the log-condition, property (20), the Hölder inequality and estimate (23), we get successively

\[
|B(x, r)|^{\frac{\alpha(x)}{n} - 1} \int_{B(x, r)} |f(y)| dy = |B(x, r)|^{\frac{\alpha(x)}{n} - 1} \int_{\tilde{B}(x, r)} r^{\frac{\lambda(x)}{p'(|\cdot|)}} r^{-\frac{\lambda(x)}{p(|\cdot|)}} |f(y)| dy

\leq C |B(x, r)|^{\frac{\alpha(x)}{n} - 1} \int_{\tilde{B}(x, r)} r^{-\frac{\lambda(x)}{p(|\cdot|)}} |f(y)| dy

\leq C r^{\alpha(x) - n + \frac{\lambda(x)}{p(|\cdot|)}} \left\| r^{-\frac{\lambda(x)}{p(|\cdot|)} f \chi_{\tilde{B}(x, r)}} \right\|_{p(|\cdot|)} \left\| \chi_{\tilde{B}(x, r)} \right\|_{p'(|\cdot|)}

\leq C r^{\alpha(x) - n + \frac{\lambda(x)}{p(|\cdot|)}} \left\| f \right\|_{p(|\cdot|),\lambda(|\cdot|)} \left\| B(x, r) \right\|_{p(|\cdot|)}^{-1} \leq C \|f\|_{p(|\cdot|),\lambda(|\cdot|)}.
\]

**Remark 2.** Since the set \( \Omega \) is bounded, inequality (43) obviously holds also in the super-critical case \( p(x) > \frac{n-\lambda(x)}{\alpha(x)} \).

In the limiting situation, the mapping properties of the Riesz potential operator \( I^{\alpha(\cdot)} \) and the fractional maximal operator \( M^{\alpha(\cdot)} \) are slightly different. For constant exponents, it is well known that a result similar to Theorem 5 holds for
\( I^\alpha \) only if we replace \( L^\infty(\Omega) \) by the space \( BMO = \{ f : M^f \in L^\infty \} \) equipped with the norm
\[
\| f \|_{BMO} := \| M^f \|_{\infty}
\]
(supposing that we do not distinguish functions differing by a constant).

The similar \( L^{p(\cdot),\lambda(\cdot)} \rightarrow BMO \)-boundedness holds also in the variable exponent setting. In the case of constant exponents this was proved by S. Spanne and published in [19] (see Theorem 5.4 in [19]). To extend this boundedness to the case of variable \( p(\cdot) \) and \( \lambda(\cdot) \) we make use of the pointwise estimate (see [1], Proposition 3.3)
\[
M^f(I^\alpha f)(x) \leq c M^\alpha f(x), \quad x \in \Omega.
\] (44)

Then from (44) and Theorem 5 the following statement follows immediately.

**Theorem 6.** Let \( \lambda(x) \geq 0, \ 0 < \alpha < n, \ \sup_{x \in \Omega} \lambda(x) < n - \alpha \), and let \( p(x) = \frac{n - \lambda(x)}{\alpha} \). Then under condition (9) the operator \( I^\alpha \) is bounded from \( L^{p(\cdot),\lambda(\cdot)}(\Omega) \) to \( BMO(\Omega) \).

**Remark 3.** Within the framework of variable exponent spaces the result of Theorem 6 seems to be new even in the case of variable Lebesgue spaces, that is, in the case, where \( \lambda(x) \equiv 0 \).

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