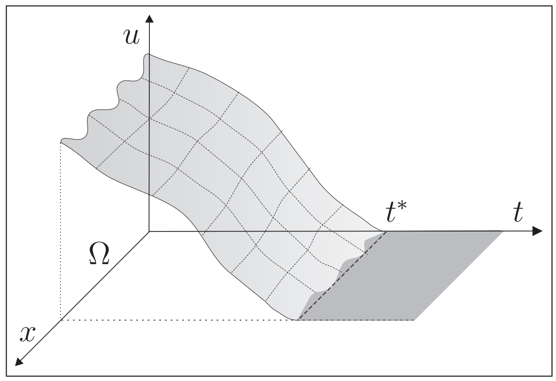

Localização de Soluções para Equações de Navier-Stokes Planares

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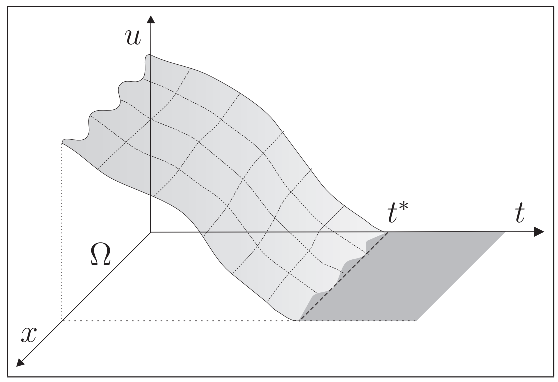
Tese para a obtenção do Grau de Doutor em Matemática,
Especialidade de Análise Matemática

Faro, Janeiro de 2004



Localization of Solutions for Planar Navier-Stokes Equations

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Título: Localização de soluções para equações de Navier-Stokes planares.

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Resumo

Consideramos escoamentos de fluidos viscosos incompressíveis, em faixas semi-infinitas horizontais, governados pelos sistemas estacionários de Stokes, Navier-Stokes e Boussinesq. Nos problemas de Stokes [6, 7, 8] e Navier-Stokes [9, 14], consideramos condições de fronteira nulas nas paredes horizontais, velocidades possivelmente não nulas nas entradas das faixas e consideramos velocidades nulas no infinito. Para o problema de Boussinesq [10], consideramos o sistema formado pelas equações de Navier-Stokes, com as condições de fronteira anteriormente mencionadas, e pela equação estacionária para a temperatura, com temperatura possivelmente não nula na fronteira compacta e temperatura nula no infinito. Mostramos como estes fluidos podem ser parados a uma distância finita das entradas das faixas por meio de um campo de forças dissipativo com memória dependendo de um modo sublinear da velocidade. Consideramos também um escoamento planar de um fluido viscoso incompressível, num domínio resultante do produto de uma faixa semi-infinita horizontal com o intervalo de tempo $(0, \infty)$, governado pelo sistema de Navier-Stokes evolutivo [11, 13]. Neste caso, consideramos condições de fronteira nulas na fronteira compacta, velocidade nula no infinito e uma velocidade inicial possivelmente não nula. Para este caso, mostramos como este fluido pode ser parado num tempo finito. Todas esta propriedades são denominadas por efeitos de localização e são demonstradas reduzindo os problemas considerados a outros, não lineares do tipo bi-harmónico, para os quais a localização das soluções é obtida por aplicação de um método de energia apropriado. Como a presença dos termos não lineares definidos através dos campos de forças não é habitual em literatura de Mecânica dos Fluidos, estabelecemos também alguns resultados de existência e unicidade de soluções fracas para estes problemas. Finalmente, fazemos uma incursão da aplicação dos nossos resultados em Elasticidade Clássica, Hidrodinâmica Magnética e Escoamentos Quase-Geostróficos.

Palavras-chave

Domínios planares, sistema de Stokes estacionário, sistema de Navier-Stokes estacionário, sistema de Boussinesq estacionário, sistema de Navier-Stokes evolutivo, campo de forças dissipativo com memória, efeitos de localização, métodos de energia.

Title: Localization of solutions for planar Navier-Stokes equations.

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Ph. D.: Branch of Mathematics, speciality of Mathematical Analysis.

Supervisor: Stanislav Nikolaevich Antontsev, Full Professor.

Abstract

We consider planar flows of incompressible viscous fluids, in semi-infinite horizontal strips, governed by the stationary Stokes, Navier-Stokes and Boussinesq systems. In the Stokes [6, 7, 8] and Navier-Stokes [9, 14] problems, we consider zero boundary conditions on the lateral walls, possible non-zero velocities at the strip entrances and we prescribe zero velocities at infinity. For the Boussinesq problem [10], we consider the Navier-Stokes equations supplemented with the aforementioned boundary conditions, and we consider the coupled stationary equation for the temperature added with a possible non-zero temperature on the compact boundary and with a prescribed zero temperature at infinity. We show how these fluids can be stopped at a finite distance from the strip entrances by means of feedback dissipative fields depending in a sublinear way on the velocity field. We consider also a planar flow of an incompressible viscous fluid, in a domain resulting from the product of a horizontal semi-infinite strip with the time interval $(0,\infty)$, governed by the evolutionary Navier-Stokes [11, 13] system. In this problem, we consider zero boundary conditions on the compact boundary, zero prescribed velocity at infinity and a possible non-zero initial velocity. For this case, we show how this fluid can be stopped in a finite time. All these properties are denoted as localization effects and are proved reducing the considered problems to non-linear bi-harmonic types for which the localization of solutions is obtained by means of the application of a suitable energy method. Since the presence of the non-linear terms defined through the body forces fields is not standard in the Fluid Mechanics literature, we establish also some results about existence and uniqueness of weak solutions for these problems. Finally, we make an attempt to apply these results in Classical Elasticity, Magneto-Hydrodynamics and Quasi-Geostrophic Flows.

Keywords

Planar problems, stationary Stokes system, stationary Navier-Stokes system, stationary Boussinesq system, evolutionary Navier-Stokes system, feedback dissipative forces field, localization effects, energy methods.

"Ainda que eu tenha o dom da profecia e conheça todos os mistérios e toda a ciência, ainda que eu tenha tão grande fé que transporte montanhas, se não tiver amor, nada sou."

1 Cor 13, 2.

"And though I have the gift of prophecy, and understand all mysteries and all knowledge, and though I have all faith, so that I could remove mountains, but have not love, I am nothing."

1 Cor 13, 2.

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" Muito obrigado! ø Muchas gracias! Thank you very much!"

H.B. de Oliveira.

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I thank all my colleagues at University of Algarve for all the help and support they gave me. Among these, I want to thank personally my colleague M.T. Alzugaray for having introduced me to Professor S.N. Antontsev, to B. Bird for some help he gave me with the English, to C. Coelho for some suggestions he gave me in how to improve LaTeX, to Professor V. Kravchenko as the organizer of the weekly seminars on Mathematical Analysis and Applications where I could present some of my work, and especially to Professor S. Samko for his disposition in helping me to learn subjects of Functional Analysis and others, and for having corrected some part of this text. I am deeply grateful to him, because in a sense he was also a co-supervisor to me.

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This work was developed in many places and with the help of many institutions. Moreover it required lots of bibliography, only available in scientific libraries. I am deeply grateful to all the facilities and the support that I receive by many people working in those places. I will only mention the names of institutions and libraries, but my thanks extend to all the people working there who helped me. My thanks to the Mathematics Department and the Faculty of Sciences and Technology at University of Algarve, the Mathematics Department at University of Beira Interior, the Faculty of Mathematics at Complutense University of Madrid, the Center of Mathematics and Fundamental Applications at University of Lisbon, the Portuguese Foundation for Science and Technology, the Central Library at University of Algarve, the Library of Interdisciplinary Complex at University of Lisbon, the Libraries of Technical Superior Institute at Technical University of Lisbon, the Central Library at University of Beira Interior and the Central Library of the Faculty of Sciences at University of Lisbon.

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Faro, January 2004.
Hermenegildo Borges de Oliveira.

Preface

"Open your freshly published book at random, the first thing you will see is a mistake."

K.O. Friedrichs [47, p. 10].

The author has chosen to start the preface to this text with these words of that great mathematician K.O. Friedrichs (1901-1982), not to excuse himself for any mistake that might have been left, in spite of all the corrections that have been made, but only to constat an evidence. In this kind of work there is always some errors that persist. What we hope is that the mistakes that might appear are not of scientific nature, because those are the worst in a pure science such as Mathematics.

This work started with my collaboration with Professor S.N. Antontsev that I established still during the year of 2000. In our first meeting, Professor S.N. Antontsev proposed that I choose one of several problems to research in the area of Partial Differential Equations which model physical phenomena from Fluid Mechanics. After spending some time studying these problems, I have chosen the problem consisting in looking for a suitable forces field that could have the property to stop a fluid governed by the incompressible homogeneous[1] Navier-Stokes equations in a semi-infinite strip, with a possible non-zero velocity at the strip entrance and prescribed zero velocity at infinity, and with an appropriated initial condition for the evolutionary problem[2]. We have found that such a forces field must be a feedback nonlinear dissipative field and that we can establish the same kind of properties for a great variety of equations governing fluid flows.

Although the material in this work is almost entirely self-contained, it will be more easier for the readers who already possess a previous knowledge of some subjects. The reader should be familiar with Functional Analysis, Differential Equations and Continuum Mechanics. In Functional Analysis, the reader should know Lebesgue and Sobolev function spaces and the main results in these subjects as well. Some knowledge of Partial and Ordinary Differential Equations is also necessary, specifically in elliptic and parabolic differential equations of any order and in any dimension, and in ordinary differential inequalities. The cornerstone of this work is Continuum Mechanics, and

[1]The results presented in this work concern only with incompressible homogeneous fluids. Thus, when no confusion can be made, we will drop the adjectives incompressible and homogeneous.

[2]During this work we will use the words evolutionary and time dependent problem with the same meaning, a problem for which the unknown functions depend, not only on the space variable, but also on an extra variable called time.

thus a particular knowledge of this subject is necessary, especially of Fluid Mechanics and with its mathematical treatment.

This thesis is organized in four main chapters which treat the stationary Stokes problem in Chapter 2, the stationary Navier-Stokes problem in Chapter 3, the stationary Boussinesq problem in Chapter 4 and the evolutionary Navier-Stokes problem in Chapter 5. Also an introduction is presented in Chapter 1 of those subjects of main concern in the four subsequent chapters. Some possible directions of applications of our results are presented in Chapter 6. The conclusions of this work are in Chapter 7, and where is given some other research projects in the forthcoming years. Finally there is an appendix where the used notation is referred and the function spaces present in the text are introduced.

Chapter 1 is devoted to the review of some results, in Section 1.1, that will be used in the sequel and to introduce the reader to the main subjects that are treated in this work. Those are localization effects in Section 1.2, energy methods in Section 1.3 and Navier-Stokes equations in Section 1.4.

In Chapter 2, we deal with the simplest problem, the stationary Stokes problem for a homogeneous incompressible fluid. We consider this problem in a semi-infinite strip with zero velocity at the horizontal walls, non-zero velocity at the strip entrance and prescribed zero velocity at infinity. In Section 2.1 we derive these planar Stokes equations from the Navier-Stokes equations presented in Section 1.4. Then, in Section 2.2 we give a precise statement of the problem under consideration and we present the forces field that will be considered. The motivation to consider such a forces field is given in Section 2.3, where we made a historical summary of the results obtained by other authors in the field of Partial Differential Equations and which lead us to obtain the desired localization effect. In Section 2.4 our problem is presented in a more rigorous mathematical form, as well the framework to prove the existence and the uniqueness of weak solution is provided. For these results, the collaboration established with Professor J.I. Díaz in the meanwhile was fundamental. The next two sections, Sections 2.5 and 2.6, are devoted to establish the localization effects which are denoted there by stopping effect and stagnation effect. In the last section, Section 2.7, some generalizations of our results are given, specifically by considering localized forces field in the sense that they only act until a finite distance, thought big enough, from the strip entrance.

The results of Chapter 2 are extended in Chapter 3 for the stationary Navier-Stokes problem in the same domain and with the same boundary conditions. We start also by an introductory section, Section 3.1, where is explained how the equations treated in this chapter are derived from the Navier-Stokes equations presented in Section 1.4 and where the complete statement of the problem is given. In Section 3.2 the existence and uniqueness of weak solution is proved. We prove the localization effects in Section 3.3, with the help of a result proved in an appendix, in Section 3.4.

In Chapter 4 we consider a non-standard stationary Boussinesq problem. We introduce the problem in Section 4.1, where we recall the derivation of the Boussinesq approximation, and in Section 4.2 we give its precise statement. Some results on the existence and uniqueness of weak solutions are proved in Section 4.3. In Section 4.4 we establish the localization effect for the velocity and, in consequence of that, we prove, in Section 4.5, that the temperature has exponential decay.

The last of main chapters of this thesis is Chapter 5 where the evolutionary Navier-Stokes equations are considered. The results presented in this chapter do not answer completely to all the questions we would like. In fact we are still working on this problem. Nevertheless, many results on existence and uniqueness of a weak solution, as well the localization effect in time are already possible to show. We present these results either in their complete rigorous form, the case of localization effects in time, or just as simple statements, the case of existence and uniqueness of a weak solution, whose proofs are addressed to the article in preparation by Antontsev et al [11]. The problem is presented in Section 5.1 and in Section 5.2 is given its weak formulation. In Section 5.3 several localization effects in time for planar bounded or unbounded domains are proved. The last section, Section 5.4, deals with the Cauchy problem, where, under certain conditions, some localizations effects in time are also proved.

In Chapter 6 we point out the resemblance between our results and some possible applications. During our research, we have found in the literature, or have just heard from some people working in the applications, as Professors J.I. Díaz, V. Kalantarov, A.V. Kazhikov and J.L. Vasquez, that our results could be applied in some situations of physical interest. In this chapter we make an attempt to use our results towards the applications in Classical Elasticity in Section 6.1, Magneto-Hydrodynamics in Section 6.2 and Quasi-Geostrophic Flows in Section 6.3.

The last chapter, Chapter 7, is dedicated to the conclusions of our work that, at this moment, can be made. Rather than concluding anything, we put many questions that have arisen to us during this work. We also point out some work which we are developing at the moment and others queries that we would like to answer positively. Of course there are also many questions the reader can pose and we would be glad and grateful if the reader address them to us.

Many of the results of this thesis were presented in several seminars given by the author at University of Algarve, University of Beira Interior, University of Évora, Technical Superior Institute at Technical University of Lisbon, Complutense University of Madrid and in the scope of the project "Nonlinear Partial Differential Equations and Free Boundary Problems". The author has also presented communications of some of these results in the International Congress "Navier-Stokes Equations and Related Topics (NSEC8)" held in 2002 at St. Petersburg, Russia, in the "Winter School on NonLinear Partial Differential Equations" held in 2003 at Technical Superior Institute, Lisbon, and in the International Conference "Nonlinear Partial Differential Equations" held in 2003 at Alushta, Ukraine. From these communications and others the abstracts [6, 13, 14] were published. Article [7] was also published and articles [8, 9, 10] were accepted for publication and will appear as soon as these journal and proceedings can it.

Faro, January 2004.
Hermenegildo Borges de Oliveira.

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Chapter 1

Introduction

In this chapter we introduce the subjects which will be the aim of our work. Section 1.1 is devoted to review some results that will be used in the sequel. The presentation refers to the main bibliography we have used. In Section 1.2 we define the localization effects which will be studied in this work. Here, we give also the two main methods available to carry out this study: super and subsolutions method and energy methods. The presentation is essentially based in the monographs by Antontsev et al [12], Díaz [34] and Èl'sgol'c [42]. In Section 1.3 a specific energy method which will be used in the forthcoming chapters to obtain the desired localization effects is described. Besides the bibliography cited in this section, we must mention the monograph by Galdi and Rionero [52] which we also have read. Navier-Stokes equations are derived in Section 1.4 from the principles of conservation of mass, linear momentum and energy. This derivation is made in the language of modern Continuum Mechanics and, therefore, we need to distinguish Newtonian from non-Newtonian fluids. Besides the bibliography cited in this section, many other sources have been seen. We would like only to mention the monographs by Antontsev et al [15], Feistauer [45], Galdi [49], Kane and Sternheim [65], Kundu [71], and the survey by Temam [116].

1.1 Preliminaries

"The maximum principle is an important feature of second order elliptic equations that distinguish them from equations of higher order and systems of equations."

D. Gilbarg and N.S. Trudinger [55, p. 32].

In this section, we recall some known results that will be used in the sequel. These results extend from elementary inequalities to more deep results from Functional Analysis and Measure Theory.

For every $a,\, b \geq 0$ and $\alpha,\, \beta \geq 0$, the following *algebraic inequality* holds

$$a^{\alpha} b^{\beta} \leq (a+b)^{\alpha+\beta}. \tag{1.1.1}$$

The *Cauchy-Schwarz inequality*

$$|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$$

holds for every $\mathbf{x},\, \mathbf{y} \in \mathbb{R}^N$.

The *Young inequality*

$$ab \leq \frac{1}{p}(\varepsilon\, a)^p + \frac{1}{p'}\left(\frac{b}{\varepsilon}\right)^{p'}, \quad \frac{1}{p}+\frac{1}{p'}=1$$

holds for every $a,\, b \geq 0$, $\varepsilon > 0$ and $1 < p < \infty$. If we take $\varepsilon = \sqrt[p]{\epsilon\, p}$, then we obtain the equivalent Young inequality $ab \leq \epsilon\, a^p + C(\epsilon)\, b^{p'}$, with $C(\epsilon) = 1/[(\epsilon\, p)^{p'/p} p']$. For $p = 2$, this inequality is called the *Cauchy inequality.*

Let Ω be a subdomain of $\mathbb{R}^N$ with a Lipshitz compact boundary $\partial\Omega$. Then, the *Gauss-Green Theorem* asserts that the unit outward normal $\mathbf{n}$ exists almost everywhere on $\partial\Omega$ and

$$\int_{\Omega} u_{x_i} d\,\mathbf{x} = \int_{\partial\Omega} u\, \mathbf{n}_i\, d\mathbf{s}, \quad i = 1, \ldots, N,$$

for every $u \in \mathrm{H}^1(\Omega)$. In consequence the *integration-by-parts formula*

$$\int_{\Omega} u_{x_i} v\, d\mathbf{x} = \int_{\partial\Omega} u\, v\, \mathbf{n}_i\, d\mathbf{s} - \int_{\Omega} u\, v_{x_i}\, d\mathbf{x}, \quad i = 1, \ldots, N,$$

is valid for every $u,\, v \in \mathrm{H}^1(\Omega)$. Moreover, the *Green formula*

$$\int_{\Omega} u \bigtriangleup v\, d\mathbf{x} = \int_{\partial\Omega} u\, \nabla v\cdot\mathbf{n}\, d\mathbf{s} - \int_{\Omega} \nabla u\cdot\nabla v\, d\mathbf{x}$$

holds for every $u \in \mathrm{H}^1(\Omega)$ and $v \in \mathrm{H}^2(\Omega)$.

If $p > 1$, $r \neq 1$ and $F(x)$ is defined by $F(x) = \int_0^x f(t)\, dt$ for $r > 1$ and $F(x) = \int_x^\infty f(t)\, dt$ for $r < 1$, with $f(x) \geq 0$, then the *Hardy inequality*

$$\int_0^\infty x^{-r} F(x)^p\, dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(x\, f(x))^p\, dx \tag{1.1.2}$$

holds, unless $f \equiv 0$. The constant is the best possible and if $p = 1$, the two sides of (1.1.2) are equal (cf. Hardy et al [60, Theorem 330]).

Let Ω be a subdomain of $\mathbb{R}^N$ and assume $1 \leq p \leq \infty$. Then the *Hölder inequality*

$$\int_\Omega |u\, v|\, d\mathbf{x} \leq \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^{p'}(\Omega)}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

holds for every $u \in \mathrm{L}^p(\Omega)$ and $v \in \mathrm{L}^{p'}(\Omega)$.

Let Ω be a subdomain of $\mathbb{R}^N$ bounded at least in one direction. Assume $1 \leq p < \infty$, then the *Poincaré inequality*[1]

$$\int_\Omega |u|^p\, d\mathbf{x} \leq C \int_\Omega |\nabla u|^p\, d\mathbf{x}, \quad C = C(p, |\Omega|), \tag{1.1.3}$$

holds for every $u \in \mathrm{W}_0^{1,p}(\Omega)$, and, if Ω is unbounded, $|\Omega|$ stands for the maximum width of Ω in the bounded direction (cf. Gilbarg and Trudinger [55]).

In this work we will appeal on many occasions to the *Weak Maximum Principle.* For many purposes it suffices to have the *Classical Weak Maximum Principle.* But, here, we consider its natural extension to operators in divergence form and for functions in the Sobolev space $\mathrm{H}^1(\Omega)$.

Theorem (Weak Maximum Principle) *Let us consider the following second order linear elliptic*[2] *operator having its principal part in divergence form*

$$L(u) = \mathrm{div}(\mathbf{a}\nabla u + \mathbf{b}u) + \mathbf{c} \cdot \nabla u + d\, u, \tag{1.1.4}$$

whose coefficients $\mathbf{a}$ *(matrix),* $\mathbf{b}$, $\mathbf{c}$ *(vectors) and* d *(scalar) are assumed to be measurable and bounded in a domain* Ω *of* $\mathbb{R}^N$*. If* $u \in \mathrm{H}^1(\Omega)$ *satisfies* $L(u) \geq 0$ *(*≤ 0*) in* Ω*, then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u_+ \quad \left(\textit{resp.}\ \inf_\Omega u \geq \inf_{\partial\Omega} u_- \right).$$

Notice that for the classical weak maximum principle, the condition it is imposed that the coefficient of u (in this case, $\mathrm{div}\,\mathbf{b} + d$) is non-positive. But since the derivatives $\mathrm{div}\,\mathbf{b}$ need not exist as functions, the non-positivity of $\mathrm{div}\mathbf{b} + d$ must be interpreted in a generalized sense, *i.e.*

$$\int_\Omega (\mathrm{div}\,\mathbf{b} + d) v\, d\mathbf{x} \leq 0, \quad \text{for every } v \in \mathrm{C}_0^1(\Omega).$$

Since $\mathbf{b}$ and d are bounded, this inequality will continue to hold for all non-negative u in $\mathrm{W}_0^{1,1}(\Omega)$ (cf. Gilbarg and Trudinger [55]).

---

[1] This is a weaker form of a Sobolev inequality and the true Poincaré inequality corresponds to the case $p = 2$. There are also some authors that, in the case of $p = 2$, refer to this inequality as the Friedrichs inequality (see Nečas [90]).

[2] Ellipticity means that the coefficient matrix $\mathbf{a}$ is positive definite in the domain of the respective arguments.

Strongly related with the Weak Maximum Principle is the next result we present, usually denoted as the *Comparison Principle.*

Theorem (Comparison Principle) *Let u_1, $u_2 \in \mathrm{H}^1(\Omega)$ satisfy $L(u_1) \geq L(u_2)$ in Ω and corresponding to boundary conditions h_1 and h_2, respectively. If $h_1 \leq h_2$ on $\partial\Omega$, then $u_1 \leq u_2$ in Ω.*

The next result gives us suitable interpolation inequalities whose applications are of the utmost importance in this thesis. Here, we have considered the original references by Gagliardo [48] and Nirenberg [91].

Theorem (Gagliardo-Nirenberg) *Assume $\Omega \subset \mathbb{R}^N$ and let j, k, $l \in \mathbb{Z}$ with $0 \leq j < k$, $k \geq 1$ and $1 \leq q,\ r < \infty$.*
(i) If Ω is unbounded, then

$$\left\|\mathrm{D}^j u\right\|_{\mathrm{L}^p(\Omega)} \leq C \left\|\mathrm{D}^k u\right\|^{\theta}_{\mathrm{L}^q(\Omega)} \|u\|^{1-\theta}_{\mathrm{L}^r(\Omega)}, \tag{1.1.5}$$

where θ is given by

$$\frac{1}{p} = \frac{j}{N} + \theta\left(\frac{1}{q} - \frac{k}{N}\right) + (1-\theta)\frac{1}{r}, \quad \textit{for all} \quad \frac{j}{k} \leq \theta \leq 1$$

and $C = C(j,k,N,p,q,r,\theta)$.
(ii) If Ω is bounded, then

$$\left\|\mathrm{D}^j u\right\|_{\mathrm{L}^p(\Omega)} \leq C_1 \left\|\mathrm{D}^k u\right\|^{\theta}_{\mathrm{L}^q(\Omega)} \|u\|^{1-\theta}_{\mathrm{L}^r(\Omega)} + C_2 \|u\|_{\mathrm{L}^{\tilde{q}}(\Omega)}, \quad \textit{for any } \tilde{q} \geq 1, \tag{1.1.6}$$

where $C_i = C_i(j,k,N,p,q,r,\theta,\Omega)$, $i = 1,2$. We remark that (1.1.5) and (1.1.6) only make sense if its right-hand sides are finite.

The following two results give us imbeddings of Sobolev spaces. Here, we have followed Adams [1]. The first one, is the so-called *Sobolev Imbedding Theorem.*

Theorem (Sobolev) *Let Ω be a domain in $\mathbb{R}^N$ having the cone property[3] and let Ω^k be the k-dimensional domain obtained by intersecting Ω with a k-dimensional plane in $\mathbb{R}^N$, with $1 \leq k \leq N$. Let j, m be non-negative integers and let $1 \leq p < \infty$.*
PART I. If $mp < N$, the following imbedding holds:

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{W}^{j,q}(\Omega^k) \quad \textit{if} \quad N - mp < k \leq N \quad \textit{and} \quad p \leq q \leq kp/(N-mp).$$

PART II. If $mp = N$, then the following imbedding holds:

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{W}^{j,q}(\Omega^k) \quad \textit{if} \quad p \leq q < \infty.$$

[3] A domain Ω has the cone property, if each point of Ω is the vertex of a finite cone contained in Ω along with its closure.

PART III. If $mp > N$, then the following imbedding holds:

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{C}^j(\overline{\Omega}).$$

If Ω is an arbitrary domain in $\mathbb{R}^N$, these imbeddings hold provided $\mathrm{W}^{j+m,p}(\Omega)$ is replaced by $\mathrm{W}_0^{j+m,p}(\Omega)$. Moreover, if Ω^k has finite volume, the imbeddings of Parts I and II also hold for $1 \leq q < p$.

The second result is also related to Sobolev, and is usually called the *Sobolev Compact Imbedding Theorem.* But contrary to the theorem name, the imbeddings in these theorems are due to F. Rellich (case $p = 2$) and V. Kondrachov (arbitrary p).

Theorem (Rellich-Kondrachov) *Let Ω be a domain in $\mathbb{R}^N$ having the cone property, Ω_0 a bounded subdomain of Ω and Ω_0^k the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^N$. Let j, $m \in \mathbb{Z}$ such that $j \geq 0$, $m \geq 1$ and let $1 \leq p < \infty$.*
PART I. If $mp \leq N$, then the following imbeddings are compact:

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{W}^{j,q}(\Omega_0^k) \quad \textit{if} \quad 0 < N - mp < k \leq N \quad \textit{and} \quad 1 \leq q \leq kp/(N - mp);$$

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{W}^{j,q}(\Omega_0^k) \quad \textit{if} \quad N = mp,\ 1 \leq k \leq N \quad \textit{and} \quad 1 \leq q < \infty.$$

PART II. If $mp > N$, then the following imbeddings are compact:

$$\mathrm{W}^{j+m,p}(\Omega) \to \mathrm{C}^j(\overline{\Omega_0}) \quad \textit{and} \quad \mathrm{W}^{j+m,p}(\Omega) \to \mathrm{W}^{j,q}(\Omega_0^k) \quad \textit{if} \quad 1 \leq q < \infty.$$

If Ω is an arbitrary domain in $\mathbb{R}^N$, these imbeddings are compact provided $\mathrm{W}^{j+m,p}(\Omega)$ is replaced by $\mathrm{W}_0^{j+m,p}(\Omega)$.

Now, we recall some results about fixed point theorems, where we have followed Gilbarg and Trudinger [55].

Theorem (Schauder) *Let A be a compact convex set in a Banach space* B *and let T be a continuous mapping of A into itself. Then T has a fixed point, that is, $T\,x = x$ for some $x \in$* A.

Theorem (Leray-Schauder) *Let B be a Banach space and let T be a compact mapping*[4] *of $\mathrm{B} \times [0,1]$ into* B *such that $T(x, 0) = 0$ for all $x \in \mathrm{B}$. Suppose there exists a constant M such that $\|x\|_{\mathrm{B}} < M$ for all $(x, \lambda) \in \mathrm{B} \times [0,1]$ satisfying $x = T(x, \lambda)$. Then the mapping T_1 of* B *into itself given by $T_1(x) = T(x, 1)$ has a fixed point.*

The final part of this section is devoted to some results from Measure Theory. For these results, we have followed Dunford and Schwartz [40], but restrict ourselves to the case of Lebesgue measure spaces.

[4]A continuous linear mapping between two Banach spaces is called compact or completely continuous, if the images of bounded sets are precompact, their closures are compact.

Theorem (Vitali) *Let* $1 \le p < \infty$ *and* f_n *be a sequence of functions in* $\mathrm{L}^p(\Omega)$ *converging almost everywhere to a function* f. *Then* f *is in* $\mathrm{L}^p(\Omega)$ *and*

$$\|f_n - f\|_{\mathrm{L}^p(\Omega)} \to 0 \quad \textit{as} \quad n \to \infty$$

if and only if

$$\lim_{|\mathrm{E}| \to 0} \int_{\mathrm{E}} |f_n(x)|^p \, dx = 0 \quad \textit{uniformly in } n$$

and for each $\varepsilon > 0$, *there is a set* $\mathrm{E}_\varepsilon \subset \Omega$ *such that* $|\mathrm{E}_\varepsilon| < \infty$ *and such that*

$$\int_{\Omega \setminus \mathrm{E}_\varepsilon} |f_n(x)|^p \, dx < \varepsilon, \quad n = 1,\ 2,\ \ldots.$$

Theorem (Lebesgue) *Let* $1 \le p < \infty$ *and* f_n *be a sequence of functions in* $\mathrm{L}^p(\Omega)$ *converging almost everywhere to a function* f. *Suppose that there exists a function* g *in* $\mathrm{L}^p(\Omega)$ *such that* $|f_n(x)| \le |g(x)|$ *almost everywhere. Then* f *is in* $\mathrm{L}^p(\Omega)$ *and* $\|f_n - f\|_{\mathrm{L}^p(\Omega)}$ *converges to zero.*

Corollary *Let* f_n *be a monotone increasing sequence of non-negative real measurable, but not necessarily integrable, functions converging almost everywhere to a function* f. *Then*

$$\lim_{n \to \infty} \int_\Omega f_n(x) \, dx = \int_\Omega f(x) \, dx.$$

Lemma (Fatou) *Let* f_n *be a sequence of non-negative measurable, but not necessarily integrable, functions. Then*

$$\int_\Omega \liminf_{n \to \infty} f_n(x) \, dx \le \liminf_{n \to \infty} \int_\Omega f_n(x) \, dx.$$

1.2 Localization Effects

> *"Qualitative methods in mathematics are methods which make it possible in absence of a quantitative solution of a mathematical problem to indicate a number of qualitative properties of the desired solution."*
>
> L.È. Èl'sgol'c [42, p. vii].

Quantitative physical laws are idealizations of reality and, as knowledge grows, we observe that a given physical situation can be idealized mathematically in a number of different ways. It is therefore important to characterize those reasonable ideal formulations. Hadamard [59] asserts that a given problem for a partial differential equation is well-posed, if the problem in fact has a solution, this solution is unique and it depends continuously on the data given in the problem. These criteria are reasonable from the Physics point of view in most cases. Existence and uniqueness are an affirmation of the Principle of Determinism[5], without which experiments could not be repeated with the expectation of consistent data. The continuous dependence criteria is an expression of the stability of the solution, i.e., a small change in any of the problem data should produce only a correspondingly small change in the solution. These criteria are usually referred as the *qualitative properties* of the problem in contrast with the *quantitative properties* which mainly concern with finding an exact solution or, at least, an approximated one. Sometimes the qualitative analysis of a mathematical problem is only the first step of an investigation, in which one proves the existence, estimates the number of solutions and establishes some peculiarities of the solutions, thus facilitating in the future their exact or approximated solution. However, one frequently has to deal with problems in which the question is, from the beginning, purely qualitative and the finding of an exact or approximated solution does not make it possible to answer the question as posed and frequently does not even help in finding the solution of that problem.

The main goal of this thesis is the study of some qualitative properties of the solutions of some problems arising in Fluid Mechanics and which are known in the literature by the widened name of localization effects. Roughly speaking, the localization effects are all the properties which one can prove to localize the solutions in some part of the problem domain. Localized solutions occur when the influence of the data, such as initial conditions, boundary conditions and or prescribed functions, on the behavior of solutions is restricted to the points of the domain close enough to the support of the data. The localization effects depend on the nature of the problem, specifically if it is of stationary or evolutionary type. If the problem is of stationary type, the localization effect can only be studied on the space and if it occurs, we say, solely, we have a *localized solution*.

[5]Determinism is the philosophical doctrine that claims that all behavior results from preceding events or natural causes. Scientific determinism makes a very strong assertion, that all events are in principle predictable (see Popper [95]).

Definition 1.2.1 *Let Ω be an open subset of $\mathbb{R}^N$ and let $u : \Omega \to \mathbb{R}$ be a solution, at least in a weak sense, of a given stationary boundary-value problem in Ω. We say that u is a* localized solution *if it vanishes in an open subset of Ω.*

When the problem is of evolutionary type, there are two main different localization effects: *localization in time* and *localization in space*. The localization effect in time corresponds to the stabilization of solutions in a finite time to a stationary state.

Definition 1.2.2 *Let Ω be an open subset of $\mathbb{R}^N$, $0 < T \leq \infty$ and $u : \Omega \times (0, T) \to \mathbb{R}$ a solution, at least in a weak sense, of a given initial boundary-value problem in $\Omega \times (0, T)$. We say that $u(x, t)$* stabilizes in a finite time *to a stationary state u_s, if there exists $t^* \in (0, T)$ such that for all $t \in [t^*, T)$, $u(x, t) = u_s(x)$ in Ω.*

If the stationary state is zero, we say that there is an *extinction in a finite time*. Most results in the literature of *stabilization in a finite time* to a stationary state concern the case of *extinction in a finite time*.

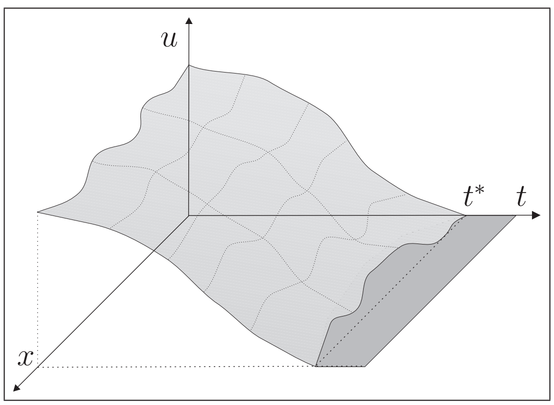

Figure 1.1: Localization effect in time.

Among the properties of stabilization in a finite time, one can find the finite speed of propagations and the formation of dead cores. Both are related to a degeneration introduced in the problem when the solution attains certain s-level, usually normalized to be zero. *Finite speed of propagations* means that the speed of propagations of disturbances from the initial data is finite. In other words we can talk about finite speed of propagations, if solutions corresponding to compact supported initial datum remain with compact support, at least for some time. *Dead core* means that, even if the initial datum is strictly positive, a region where the zero level is attained may appear in finite time. In mathematical terms this means that there exists $t^* > 0$ such that $\mathbb{R}^N \setminus \{\operatorname{supp} u(., t)\} \neq \emptyset$ for all $t \geq t^*$, in spite of $\operatorname{supp} u_0 = \mathbb{R}^N$. Even in stationary problems, one can talk about the dead core formation in the case where the solution vanishes in an interior region.

The localization effect in space corresponds to the stabilization of solutions in some space subdomain and in some interval of time.

Definition 1.2.3 *Let* Ω *be an open subset of* $\mathbb{R}^N$, $0 < T \leq \infty$ *and* $u : \Omega \times (0,T) \to \mathbb{R}$ *a solution, at least in a weak sense, of a given initial boundary-value problem in* $\Omega \times (0,T)$. *We say that* $u(x,t)$ *has the property of* localization in space, *if there exists an open subset* Ω_0 *of* Ω *and* $t_* \in (0,T]$ *such that* $u(x,t) = 0$ *a.e. in* Ω_0 *and for all* $0 \leq t < t_*$.

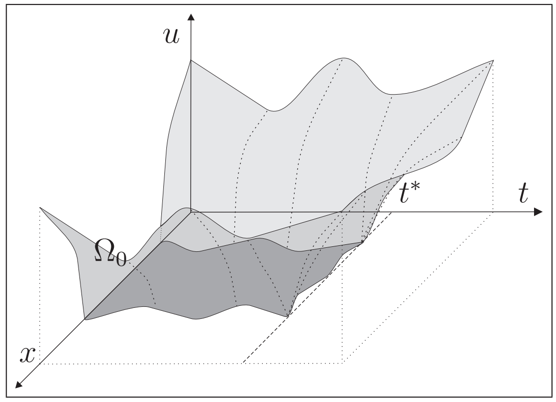

Figure 1.2: Localization effect in space.

For the sake of simplicity, we have introduced the concept of localization in space to the zero-level of $u(x,t)$. But, considering the function $u(x,t) - s$ with $s \neq 0$, we can extend this concept in a natural way with respect to the s-level. In the literature one can find two main localization effects in space: *waiting time property* and *support shrinking property*. In the special case of the right-hand side of a partial differential equation is zero ($f(x,t) = 0$), the time t_* is called the *waiting time* and then we say the solutions satisfy the *waiting time property.* The solution of some initial boundary value problem has the *support shrinking property*, if for any $t > 0$ the support of solution $u(x,t)$ is bounded, even if it is unbounded for $t = 0$.

We remark that the property of localization in time is global, *i.e.* $u(x,t)$ vanishes in the whole Ω if $t \geq t_*$, in contrast with the property of localization in space which has a local character, *i.e* $u(x,t)$ merely vanishes in Ω_0 if $t \leq t_*$. A special class of localized solutions corresponds to the case when the domain is unbounded and the supports of solutions are bounded and therefore compact.

From the mathematical point of view, all these localization effects mean that the set $\operatorname{supp} u$ is strictly contained in the problem domain. A physical meaning for this situation can be interpreted as the dissipation of the quantity under study (density, concentration, temperature, velocity, *etc.*) in some part of the domain. A good survey for applications of the localization effects to problems in Fluid Mechanics is the monograph by Antontsev et al [12].

In order to carry out the study of the localization properties, two main general methods are available: the *super and subolutions method* and the *energy methods.* The first method is based on the construction of adequate super and subsolutions $\overline{u}$ and $\underline{u}$ to which, jointly with the solution u of the considered partial differential equation, the Comparison Principle is applied. Such functions can be chosen with compact support and so, by a comparison argument, $\underline{u} \leq u \leq \overline{u}$ in the considered domain Ω, which implies that $\operatorname{supp} u$ is also a compact subset. Due to the use of Comparison Principle, this method is especially useful in the study of second and also first order equations. Although the Comparison Principle implies automatically the uniqueness of solution, this method can also be applied to some monotone problems and systems in which the uniqueness fails. The idea for using super and subsolutions to establish the localization effects is quite classical in the theory of partial differential equations and, for nonlinear problems, such functions are taken locally as suitable interior or boundary barriers functions. The use of Comparison Principle, and to the best of our knowledge, goes back to the work of Oleinik et al [92] on a degenerate parabolic equation.

The energy methods rely on the idea of finding some ordinary differential inequalities satisfied by some energy functions involving the integral, over some subsets, of some suitable chosen differential expressions of the solutions. These methods will be the scope of the next section.

An important question connected with the localization effects, is the *free boundary*, also called the *interface*, generated by the unknown boundary of the support of the solutions. For instance, the function

$$u(x) = u_0 \left(1 - \frac{x}{x_0}\right)_+^{p/(p-q-1)}, \tag{1.2.7}$$

where

$$x_0 = u_0^{(p-q-1)/p} \frac{p}{p-q-1} \left[\frac{(p-1)q}{p-q-1}\right]^{1/p}, \tag{1.2.8}$$

is a solution of the following one-dimensional free-boundary problem: to find a non-negative function $u(x)$ and a positive finite number x_0 such that u satisfies

$$L\,u \equiv -\frac{d}{d\,x}\left(|u_x|^{p-2}u_x\right) + |u|^{q-1}u = 0, \quad q < p-1, \tag{1.2.9}$$

in $(0, x_0)$ and

$$u(0) = u_0, \quad u(x_0) = 0, \quad |u_x(x_0)|^p u_x(x_0) = 0.$$

On the other hand, the function $u(x)$ given above in (1.2.7)-(1.2.8) is a localized solution of equation (1.2.9) in the domain $\Omega = (0, \infty)$ and observe that $\operatorname{supp} u = (0, x_0)$. In this way, by a *free boundary*, we mean a curve separating the regions where the solution vanishes (or stabilizes at a s-level) or it is positive (see Antontsev et al [12] and Díaz [34]).

1.3 Energy Methods

"All of the methods which lead to variational problems for bounded functionals can be considered energy methods in a generalized sense."

D.D. Joseph [63, p. 3].

By *energy method* one denotes a suitable mathematical device for deriving estimates of solutions to differential equations. The name of the method is due to the fact that it is usually founded upon conservation and balance laws from continuum mechanics and which must be satisfied by the solutions. A typical example of an energy method, and to our knowledge the first one, is given by the approach introduced by Lyapunov [83] for studying the stability of solutions to ordinary differential equations. Although found the works of Lyapunov on what we call nowadays energy methods, this mathematical device only became known with the works of Friedrichs [47] in Partial Differential Equations to establish existence and uniqueness results and to prove the convergence of some difference schemes. In Fluid Mechanics the *energy method* was introduced by Serrin [104] in a way which consists in forming the kinetic energy of perturbations to a given basic flow and in studying its behavior in time. This method was successively generalized and enlarged by Joseph and his co-workers [63] for studying nonlinear stability of viscous incompressible flows in bounded domains. Still in Fluid Mechanics, Straughan [112] has developed the energy method in a variety of contexts: half-space problems, geophysical problems, convection driven by surface tension, convection in other classes of fluids, time-dependent convection problems and has studied the connection with the there-called Lyapunov method in partial differential equations.

The main goal of this thesis is to study some qualitative properties of problems arising in Fluid Mechanics by using a specific energy method which is denoted by *energy method for free boundary problems* (cf. Antontsev et al [12]). Energy methods are of special interest in those situations in which traditional methods based on the Comparison Principle have failed. A typical example of such a situation is either a higher-order equation or a system of partial differential equations. We note that energy methods are well suited in the study of partial differential equations systems which include equations of different types frequently arising in the mathematical models of continuum mechanics. In such systems, the various unknowns (velocity, density, saturation, *etc.*) may satisfy equations of different types and need not even be defined on the same domain. Moreover, even when the Comparison Principle holds, it may be extremely difficult to construct suitable sub or super-solutions if, for instance, the equation under study contains transport terms and has either variable or unbounded coefficients or the right-hand side. The main idea of the energy methods consists in deriving and studding suitable ordinary differential inequalities for various types of energy. In typical situations, these inequalities follow from the conservation and balance laws of Continuum Mechanics. In the simplest situation, the energy functions defined through a formal procedure coincide with the kinetic and potential energy.

The here so-called energy method for free boundary problems was introduced by Antontsev [5] in order to prove that, for general multidimensional degenerate parabolic equations, the finite speed of propagation of disturbances from initial data is finite and that, for degenerate elliptic equations, the solutions defined in domains infinite in extent with respect to one of the variables vanish on a set of positive measure. A more systematic treatment of this topic was made by Díaz and Véron [38, 39]. This method was enlarged by many authors, amongst whom Bernis [18, 21] gave the general treatment for higher-order elliptic and parabolic equations. A different approach to establish several localization effects for quasi-linear parabolic equations was given by Shishkov [107, 108] by using a slightly different energy method. Roughly speaking, the energy method for free boundary problems consists in three steps. The first is to multiply the partial differential equation by a weak solution and to integrate by parts over a variable or moving subdomain. This leads to energy integrals plus other terms over the boundary of the subdomain. Sometimes it is useful also to multiply the partial differential equation by a weight which will cancel some boundary terms. In almost all cases it is considered a variable ball or a variable half-space. The choice of integrating over a ball or a half-space is related with the nature of the problem under consideration. When using balls, the boundary conditions may not have to be zero, but it is very hard to work with when considering higher-order equations, because of the boundary terms resulting from integrating by parts. The use of half-spaces is easier to handle higher-order equations, but it requires zero Dirichlet boundary conditions. The second step is to use interpolation-imbedding inequalities related to Sobolev imbedding inequalities. These two steps give us an ordinary differential inequality which is satisfied by the natural energies associated to the problem. The independent variable of this differential inequality is just the variable which labels the moving subdomain. The third step is to deduce from the resolution of this differential inequality some qualitative properties, usually denoted by localization effects, of the solutions of the original problem. This method allows us to deal with a great variety of problems formulated in a very general form, where no monotonicity assumption on the nonlinearities is required, the Comparison Principle is not invoked and no restriction on the space dimension is required.

A common fact among the different energy methods that can be used, is the reduction, sometimes by means of quite sophisticated techniques, to some ordinary differential inequality satisfied by the energy function. This inequality is very close to the following problem[6]

$$\frac{d\,h(t)}{d\,t} + a(t)\varphi(h(t)) = 0, \quad h(0) = h_0 > 0. \tag{1.3.10}$$

Once we assume

$$\Psi(\tau) = \int_0^\tau \frac{d\,s}{\varphi(s)} < \infty, \quad \text{for every } \tau > 0 \text{ and } \varphi \geq 0,$$

[6]Due to the relevance of the equation appearing in (1.3.10), Díaz [36] proposes to call this type of equations as the Torricelli-Bernoulli equations in honor of Evangelista Torricelli (1608-1647) who proposed the related law $v = \sqrt{2gh}$ for the study of the efflux of a liquid from a small orifice in the walls of a vessel, and Daniel Bernoulli (1700-1782) who proved that $-d\,h(t)/d\,t = r^2/R^2\sqrt{2gh}$, where $v = -d\,h(t)/d\,t$ is the velocity at the bottom, r and R are the radium of the circular sections at the bottom and at the top, respectively, and g is the gravitational acceleration.

$a \geq 0$ and $a \in \mathrm{L}^1_{\mathrm{loc}}(0, \infty)$, the solution is given by

$$h(t) = \Psi^{-1}\left(\left[\Psi(h_0) - \int_0^t a(\tau)\, d\tau\right]_+\right)$$

and, in particular, if

$$\Psi(h_0) - \int_0^{T_0} a(\tau)\, d\tau = 0 \quad \text{for some} \quad T_0 > 0,$$

which is certainly the case if $\int_0^t a(\tau)\, d\tau \to \infty$ as $t \to \infty$, then

$$\exists\ T_0 > 0 : h(t) \equiv 0 \ \ \forall\ t \geq T_0.$$

This extinction property also holds for solutions of the inequality

$$\frac{d\, h(t)}{d\, t} + a(t)\varphi(h(t)) \leq 0,$$

since the Comparison Principle applies in the class of non-negative solutions.

We notice that such problems as (1.3.10) are also relevant in the application of other kinds of methods as the super and subsolutions method (cf. Díaz [34]).

1.4 Navier-Stokes Equations

"Such models were not new, having occurred in philosophical or qualitative speculations for millennia past. Navier's magnificent achievement was to put these notions in concrete form that he could derive equations of motion for them."

C. Truesdell [118, p. 455].

Navier-Stokes equations were proposed in 1822 by Navier [89] on the basis of suitable molecular models. The first mathematical description of the motion of an (ideal) fluid was formulated by Euler [43] in 1755 as a statement of the *Newton's second law of motion*[7] applied to a fluid moving under an internal force known as the pressure gradient. Navier's great achievement was to include in the Euler equations the effects of attraction and repulsion between neighboring molecules. These equations were rederived by Cauchy [32] in 1828, by Poisson [94] in 1829 and in 1843 Saint-Venant [99] published a derivation of the equations on a more physical basis applied, not only to the so-called laminar flows, but also to turbulent flows. However, it was only in 1845 that, by the clarifying work of Stokes [111], these equations found a completely satisfactory justification on the basis of the continuum mechanics approach.

We recall briefly the derivation of *Navier-Stokes equations* in the language of the modern continuum mechanics. We consider the motion of a fluid that occupies at time t a domain Ω_t of the space $\mathbb{R}^3$. For the sake of simplicity, we assume that $\Omega_t = \Omega$ is independent of time, since the mathematical difficulties for moving domains tend to hide difficulties specific of the Navier-Stokes equations. In continuum mechanics, the *Lagrangian representation* of the motion consists in providing the trajectory of each particle of fluid, $\mathbf{x} = \mathbf{\Phi}(\mathbf{a}, t)$, where $\mathbf{a} = (a, b, c)$ is the position at time 0 of the particle of fluid, $\mathbf{x} = (x, y, z)$ its position at time t. Navier-Stokes equations in their most common form correspond to the *Eulerian representation* of the flow, which provides the vector fields $\mathbf{u} = (u, v, w)$ corresponding to the *velocity* of the particle of the fluid which is at $\mathbf{x}$ at instant t. We have

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{\Phi}}{\partial t}(\mathbf{a}, t),$$

and conversely we can recover the Lagrangian representation of the motion from the Eulerian one by solving the systems of ordinary differential equations

$$\frac{d\mathbf{x}_{\mathbf{a}}(t)}{dt} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x}_{\mathbf{a}}(0) = \mathbf{a} \quad (\mathbf{x}_{\mathbf{a}}(t) = \mathbf{\Phi}(\mathbf{a}, t)).$$

Equations describing fluid flow are derived on the basis of three fundamental physical principles (cf. Serrin [104]).

Principle (Conservation of Mass) *The mass of fluid in a material volume ω does not change as ω moves with the fluid.*

[7] The force $\mathbf{F}$ needed to produce an acceleration $\mathbf{a}$ is $\mathbf{F} = m\mathbf{a}$, where m is the mass of the object.

Principle (Conservation of Linear Momentum) *The rate of change of linear momentum of a material volume* ω *equals the resultant force on the volume.*

Principle (Conservation of Energy) *The rate of change of total energy of a material volume is equal to the rate at which work is being done on the volume plus the rate at which heat is conducted into the volume.*

The *continuity equation* which expresses the Principle of Conservation of Mass, reads

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{u} \equiv \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1.4.11}$$

The *equation of motion* which expresses the Principle of Conservation of Linear Momentum, reads

$$\rho\gamma \equiv \rho \frac{d\mathbf{u}}{dt} \equiv \rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = \rho \mathbf{f} + \mathbf{div}\, \mathbf{S}, \tag{1.4.12}$$

where $\rho = \rho(\mathbf{x}, t)$ is the *density*, $\gamma = \gamma(\mathbf{x}, t)$ is the *acceleration*, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ represents *external volume forces* applied to the fluid and $\mathbf{S}$ is the (Cauchy) *stress tensor*. In this thesis we use the commonly accepted definition for a *fluid* as a *Stokesian fluid*, *i.e.*, an *isotropic continuous medium* such that $\mathbf{S}$ is a continuous function of the *rate of strain tensor*

$$\mathbf{D} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right), \tag{1.4.13}$$

$\mathbf{S} = \mathbf{f}(\mathbf{D})$, and, when $\mathbf{D} = \mathbf{0}$, $\mathbf{S}$ reduces to $-p\mathbf{I}$, the case on an *ideal fluid*, where $p = p(\mathbf{x}, t)$ is the *pressure of the fluid*, and $\mathbf{I}$ is the unit matrix. The *isotropy condition* is expressed by $\mathbf{QSQ}^{-1} = \mathbf{f}\left(\mathbf{QDQ}^{-1}\right)$ for all orthogonal transformations matrices $\mathbf{Q}$, which means that there is no preferred direction either in the fluid or in space. For a fluid defined like that, one can prove, from (1.4.12), the following principle holds (cf. Serrin [104]).

Principle (Conservation of Angular Momentum) *The rate of change of angular momentum of a material volume* ω *equals the angular momentum of the resultant force on the volume.*

As a consequence of the Principle of Conservation of Angular Momentum, the stress tensor $\mathbf{S}$ is symmetric.

The *energy equation* which expresses the Principle of Conservation of Energy, reads

$$\frac{d(\rho\, e)}{dt} + \rho\, e \operatorname{div} \mathbf{u} \equiv \frac{\partial(\rho\, e)}{\partial t} + \operatorname{div}(\rho\, e\, \mathbf{u}) = \mathbf{S} : \mathbf{D} - \operatorname{div} \mathbf{q}, \tag{1.4.14}$$

where e is the specific *internal energy* per unit mass and $\mathbf{q}$ is the rate of heat transported by conduction, usually denoted by *heat flux*[8]. Occasionally a term ($\rho\, g$, g is the *heat producing capacity*) is added to the right-hand side of (1.4.14) to account for various

[8]It is more usual to call $-\mathbf{q}$ the heat flux, since $\mathbf{q} \cdot \mathbf{n} < 0$ at points where heat is entering the body (see Truesdell and Noll [119]).

other types of energy sources in the fluid, such as those resulting from chemical reaction, radiation, *etc.* Here, we follow the commonly accepted formulation and postulate that $\mathbf{q}$ is an isotropic function of the temperature and thermodynamic state and thus must be parallel to $\nabla\theta$. Whence follows the *Fourier law*

$$\mathbf{q} = -k\nabla\theta, \tag{1.4.15}$$

where θ is the *absolute temperature* and $k \geq 0$ is a scalar function called *thermal conductivity*, which in most cases, is taken to be simply a function of ρ, θ and $|\nabla\theta|$, or even a constant called the *thermal conductivity coefficient*. Hence (1.4.14) becomes

$$\frac{d(\rho\, e)}{dt} + \rho\, e \operatorname{div} \mathbf{u} \equiv \frac{\partial(\rho\, e)}{\partial t} + \operatorname{div}(\rho\, e\, \mathbf{u}) = \mathbf{S} : \mathbf{D} + \operatorname{div}(k\nabla\theta)\,. \tag{1.4.16}$$

According to the Reiner-Rivlin principle of material objectivity, the *stress tensor* in its most general form is given by

$$\mathbf{S} = -p\mathbf{I} + \phi_0\mathbf{I} + \phi_1\mathbf{D} + \phi_2\mathbf{D}^2 \tag{1.4.17}$$

where ϕ_i, $i = 0, 1, 2$, are given scalar functions of the principal invariants of the rate of strain tensor $\mathbf{D}$ (cf. Truesdell and Noll [119]).

For the so-called *Newtonian fluids*, such as liquids and gases,

$$\phi_0 = \lambda\, I_1, \quad \phi_1 = 2\mu, \quad \phi_2 = 0$$

and the stress tensor obeys the *Stokes law*

$$\mathbf{S} = -p\,\mathbf{I} + \lambda \operatorname{div} \mathbf{u}\,\mathbf{I} + 2\mu\,\mathbf{D}, \tag{1.4.18}$$

where μ is called the *shear viscosity* and λ the *bulk viscosity*. In many cases μ and λ are constants and, in that case, they are called the *first* and the *second coefficient of viscosity*, respectively. Replacing the stress tensor (1.4.18) in (1.4.12) and (1.4.16), we obtain

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = \rho\mathbf{f} - \nabla p + \nabla(\lambda \operatorname{div}\mathbf{u}) + \mathbf{div}\,(2\mu\mathbf{D}) \tag{1.4.19}$$

and

$$\frac{\partial(\rho\, e)}{\partial t} + \nabla(\rho\, e) \cdot \mathbf{u} = -p \operatorname{div}\mathbf{u} + \lambda(\operatorname{div}\mathbf{u})^2 + 2\mu\,|\mathbf{D}|^2 + \operatorname{div}(k\nabla\theta)\,. \tag{1.4.20}$$

The set of equations constituted by (1.4.11), (1.4.19) and (1.4.20) is not closed and in order to close it, it remains to describe ρ, p, e and θ. If we choose as independent variables ρ and θ, then p and e are functions of ρ and θ, *i.e.*, obey some given state equations of the following form

$$p = p(\rho, \theta), \quad e = e(\rho, \theta). \tag{1.4.21}$$

Therefore it is necessary to introduce some supplementary information of thermodynamics (cf. Batchelor [16]).

First Law of Thermodynamics *The* entropy[9] s *of a system is given by* $\theta\, d\, s = d\, e + p\, d\,(1/\rho)$.

Second Law of Thermodynamics *For any process, the total entropy of a system plus its surroundings may never decrease.*

A consequence of the Second Law of Thermodynamics asserts that $\mu \geq 0$ and $3\lambda+2\mu \geq 0$. Equations (1.4.11), (1.4.19), (1.4.20) and state equations as (1.4.21) constitute the general set of equations upon which classical hydrodynamics is based.

At this point we need to define an important intrinsic characteristic of some fluids. If the volume of any part of the fluid remains constant during the motion, we say the *fluid* is *incompressible*, which is expressed by

$$\operatorname{div} \mathbf{u} = 0. \tag{1.4.22}$$

In this case, (1.4.19) and (1.4.20) can be simplified by eliminating terms containing $\operatorname{div} \mathbf{u}$,

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = \rho \mathbf{f} - \nabla p + \mathbf{div}\,(2\mu \mathbf{D}) \tag{1.4.23}$$

and

$$\frac{\partial (\rho\, e)}{\partial t} + \nabla(\rho\, e) \cdot \mathbf{u} = 2\mu\, |\mathbf{D}|^2 + \operatorname{div}\,(k \nabla \theta)\,. \tag{1.4.24}$$

In this case, p is the *mechanical pressure*, a fundamental dynamical variable, and μ is a scalar function of the temperature satisfying $\mu \geq 0$. The energy equation (1.4.24) is separated from the system {(1.4.22), (1.4.11), (1.4.23), (1.4.24)} and is solved after the velocity and pressure are found.

Fluids which do not exhibit property (1.4.22), such as gases, are denoted *compressible fluids*. For such fluids p is the *thermodynamic pressure* and μ and λ are scalar functions of the thermodynamic state.

If the fluid flow is *isothermic*, *i.e.*, θ is constant, then energy equations (1.4.20) and (1.4.24) are usefulness. In this case, {(1.4.11), (1.4.19)} is usually denoted by the *compressible Navier-Stokes system* and {(1.4.22), (1.4.11), (1.4.23)} by the *incompressible Navier-Stokes system*.

For *non-Newtonian fluids*, the stress tensor is given by

$$\mathbf{S} = -p\,\mathbf{I} + \mathbf{F}(\mathbf{D}, \rho, \theta), \tag{1.4.25}$$

where $\mathbf{F}$ is a nonlinear function of $\mathbf{D}$. Examples of such fluids are suspensions and high molecular-weight materials. Several constitutive relations have been suggested in order to capture characteristics of non-Newtonian fluids (cf. Showalter [109], Truesdell and Noll [119]), but we will not study fluids with such behavior here. Interesting situations are those for incompressible fluids when it is assumed that

$$\phi_1 \neq const. \quad \text{and} \quad \phi_2 = 0. \tag{1.4.26}$$

[9]Variable state which characterizes the disorder level of a system.

Some authors call fluids satisfying these conditions *purely non-Newtonian* (cf. Antontsev et al [12]) and others call them *generalized Newtonian fluids* (cf. Showalter [109]). A subclass of such fluids is constituted by the *Ostwald-de Waele fluids*, also called *power type fluids*, for which

$$\phi_1 = 2(\mu + \tau \, |\mathbf{D}|^{\mathrm{q}-1}), \quad \text{for some } \tau > 0 \text{ and } \mathrm{q} > 0, \tag{1.4.27}$$

with $|\mathbf{D}| = \mathbf{D} : \mathbf{D}$. The *fluid* is called *dilatant* or *viscous-plastic* if $\mathrm{q} > 1$ and *pseudo-plastic* if $0 < \mathrm{q} < 1$. If $\mathrm{q} = 1$, we revert into the class of Newtonian fluids.

Now, we restrict ourselves to the case of incompressible (Newtonian) fluids. In this case, the continuity equation (1.4.11) implies the density ρ is constant along the trajectories of the fluid. Hence, if the *fluid* is *homogeneous*, $\rho(\mathbf{x}, 0) = \rho_0 > 0$ is independent of $\mathbf{x}$. Then, we can divide (1.4.23) and (1.4.24) by ρ_0 to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} - \nabla p + \mathbf{div}\,(2\nu \mathbf{D}) \tag{1.4.28}$$

and

$$\frac{\partial e}{\partial t} + \nabla e \cdot \mathbf{u} = 2\nu \, |\mathbf{D}|^2 + \mathrm{div}\,(k \nabla \theta)\,. \tag{1.4.29}$$

We have set $\nu = \mu/\rho_0$, the *kinematic viscosity*, and we have renamed p/ρ_0 as p and k/ρ_0 as k, where p is now called the *hydrostatic pressure*.

To conclude this section, we notice the equations describing fluid flows must be supplemented with boundary conditions characterizing the flow on the boundary of the domain occupied by the fluid and by initial conditions determining the initial state of the flow at the beginning of the time interval considered. The question of initial condition is immediately understood from the physical point of view, but the question of boundary conditions is much more delicate and would require a detailed discussion. Our ambition in this thesis is somewhat limited since we shall consider problems set in a domain Ω, with standard mathematical boundary conditions on $\partial\Omega$.

Chapter 2

Stationary Stokes Problem

"As follows from experiments, the Stokes problem describes slow flows of very viscous fluids relatively well. Moreover, The Stokes problem is also used in iterative methods for the solution of the nonlinear Navier-Stokes equations."

M. Feistauer [45, p. 509].

This chapter is concerned with the study of the planar stationary Stokes problem. This problem is introduced in a semi-infinite strip, in Section 2.1, as a simplification of the Navier-Stokes equations presented before. The complete statement of the problem considered in this chapter is given in Section 2.2, where a result about the exponential decay for the weak solutions to the classical Stokes problem is also recalled. We give, in Section 2.3, the motivation for the forces field type considered here by giving some known results in this direction and by showing the difficulties felt at the beginning to prove the localization effect. The complete weak formulation of the problem is presented in Section 2.4. There we prove the existence theorems for this problem and the uniqueness one as well. To prove these results, we will use some known results for the Stokes equations with prescribed linear forces field. In almost the situations the reader is addressed to the precise sections of the monograph by Galdi [49] to see the proofs of these results. But, in fact, many other sources have been seen, such as the monographs by Constantin and Foias [33], Ladyzhenskaya [72] and Temam [115]. The main result of this chapter is the localization effect which we have denoted by stopping effect and is proved in Section 2.5. This result is proved in a constructive way, being presented the auxiliary results as they are needed. In Section 2.6 we prove another localization effect which we have denoted by stagnation effect. Section 2.7 is devoted to generalize these localization effects for localized forces field.

2.1 Introduction

Let us consider the *stationary Stokes system*

$$-\nu\triangle\mathbf{u} = \mathbf{f} - \nabla p, \tag{2.1.1}$$

$$\operatorname{div}\mathbf{u} = \mathbf{0} \tag{2.1.2}$$

in a semi-infinite strip like domain $\Omega = (0,\infty)\times(0,L)$, $L > 0$ a positive constant, where ν is the *kinematic viscosity coefficient*. Stokes system (2.1.1)-(2.1.2) is derived from the Navier-Stokes system {(1.4.22), (1.4.11), (1.4.23)} by assuming the fluid is homogeneous, the kinematic viscosity is constant, the velocity and pressure do not depend explicitly on time and by a linearization procedure on the resulting stationary Navier-Stokes equation. Such a linearization is made under the mechanical assumption the fluid viscosity is large, *i.e.* $\nu \gg 1$, and the velocity is sufficiently small. As follows from the experiments, this is characteristic of slow flows and means that the ratio $|(\mathbf{u}\cdot\nabla)\mathbf{u}|/|\nu\triangle\mathbf{u}|$ of *inertial* to *viscous forces* is vanishingly small, so that we can disregard the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in the stationary Navier-Stokes equation. Moreover, if we assume reference length l and velocity v, this linearization amounts to assume the dimensionless *Reynolds number* $\mathcal{R} = vl/\nu$ is suitably small. The consideration of the planar domain Ω, means that we are facing a flow with only two velocity components, say $u(x,y)$ and $v(x,y)$. This corresponds to a number of cases where it is possible to introduce a Cartesian coordinate system (x,y,z) such that the quantities describing the flow are nearly independent of the variable z and the velocity component w in the direction z is negligible. We thus obtain the model of plane flow where the domain $\Omega\subset\mathbb{R}^3$ occupied by the fluid has the form of a cylinder $\omega\times(0,c)$, $\omega\subset\mathbb{R}^2$ and $c > 0$, with axis orthogonal to the plane (x,y). The flow has the same character in all planes orthogonal to the axis z and the velocity component w vanishes.

2.2 Statement of the Problem

We consider the Stokes system (2.1.1)-(2.1.2) to whom we append a possible non-zero velocity at the strip entrance

$$\mathbf{u}(0,y) = \mathbf{u}_*(y),\ \ y\in(0,L) \tag{2.2.3}$$

and zero velocity on the lateral walls

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x\in(0,\infty). \tag{2.2.4}$$

Since Ω is unbounded, we have to prescribe the velocity at infinity. We are interested in the case

$$|\mathbf{u}(x,y)| \to 0,\ \text{ as } x\to\infty \text{ and } \ y\in(0,L). \tag{2.2.5}$$

We recall that due to the incompressibility condition (2.1.2), the first component of $\mathbf{u}_*(y) = (u_*(y), v_*(y))$ must satisfy

$$\int_0^L u_*(s)ds = 0. \tag{2.2.6}$$

We also assume the compatibility conditions

$$\mathbf{u}_*(0) = \mathbf{u}_*(L) = \mathbf{0}. \tag{2.2.7}$$

The main question we shall consider here can be stated in the following terms: *can we find a vector field* $\mathbf{f}$ *such that the weak solutions of problem (2.1.1)-(2.2.5) have compact support in* Ω, i.e.,

$$\mathbf{u} = \mathbf{0} \quad \text{for every} \quad x \geq x_{\mathbf{u}}, \quad \text{for some} \quad x_{\mathbf{u}} > 0?$$

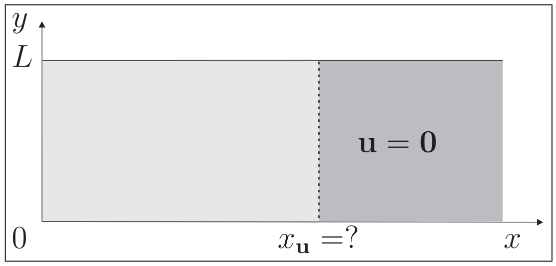

Figure 2.1: Stopping effect.

From the Physics point of view, this corresponds to search for a body forces field stopping the fluid at a finite distance from the strip entrance. This property corresponds to a localization effect as stated in Section 1.2 that we will denote by the *stopping effect*.

It is well known the weak solutions $\mathbf{u}$ of problem (2.1.1)-(2.2.5) when we prescribe zero external forces field, have an exponential decay which is optimal

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega_a)} \leq C_1 \exp(-C_2 a), \quad \text{for all} \quad a > 0, \tag{2.2.8}$$

where C_1, C_2 are positive constants depending on L, $\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}$ and ν (cf. Galdi [49, §VI.2]). This result still holds, if we consider a non-zero forces field $\mathbf{f} \in \mathbf{L}^2(\Omega)$ with compact support in Ω. In these cases, the weak solutions of problem (2.1.1)-(2.2.5) do not have compact support in Ω and the localization effect does not hold. One can easily adapt to this problem the exponential decay results on the Saint-Venant's Principle[1] in the two-dimensional linear theory of elasticity obtained separately by Toupin [117] and Knowles [68].

The answer will be positive for a body forces field given in a feedback dissipative form, $\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f} = (f_1, f_2)$, such that for every $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{u} = (u, v)$, and almost all $\mathbf{x} \in \Omega$

$$-\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \geq \delta \, |u|^{1+\sigma} - g(\mathbf{x}) \tag{2.2.9}$$

for some constants $\delta > 0$, $0 < \sigma < 1$ and some function

$$g \in \mathrm{L}^1(\Omega^{x_g}), \; g \geq 0, \; g(\mathbf{x}) = 0 \; \text{ a.e. in } \; \Omega_{x_g}, \tag{2.2.10}$$

where $\Omega^{x_g} = (0, x_g) \times (0, L)$ and $\Omega_{x_g} = (x_g, \infty) \times (0, L)$, with $0 \leq x_g < \infty$. The novelty of this forces field is the dependence of $\mathbf{f}$, not only on the spatial variable $\mathbf{x}$, but also

[1] See Remark 2.5.4 on page 41.

on the own solution $\mathbf{u}$. We say this forces field is a feedback dissipative field, because there is a dependence on $\mathbf{u}$ of its first component and it is dissipative on the second. An example of such a forces field is

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = -\delta\left(|u|^{\sigma-1}u, 0\right), \quad \sigma \in (0,1), \quad \delta > 0. \tag{2.2.11}$$

Remark 2.2.1 *We alert for the fact of the problem, such as it is stated, can be thought wrongly that we are in presence of a control problem, where, not only* $\mathbf{u}$ *and* p *are unknowns, but also* $\mathbf{f}$*. The body forces field is given satisfying (2.2.9) and (2.2.10) and furthermore, as we will see in the forthcoming sections, none technique of Control Theory is used to establish our results.*

2.3 Motivation

Compactness of the support in second order problems has been studied, in any dimension, by many authors, starting with the work of Brézis [26] on the stationary obstacle problem, although the pioneer work of Oleinik et al [92]. For instance, let us consider the following Dirichlet problem

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{\sigma-1}u = f \quad \text{in} \quad \Omega, \tag{2.3.12}$$

$$u = h \quad \text{on} \quad \partial\Omega, \tag{2.3.13}$$

where $p \geq 1$, $\sigma > 0$ and Ω is a subdomain of $\mathbb{R}^N$. The equation given in (2.3.12) appears in the study of a stationary isothermical single reaction and in the non-Newtonian stationary fluids theory (see *e.g.* Díaz [34]). When $p = 1$, (2.3.12) is a quasi-linear equation which becomes degenerate for $p \neq 2$. In that case, the equation is not *uniformly elliptic*[2] loosing its elliptic character on the set $\{x \in \Omega : \nabla u(x) = 0\}$. Existence, uniqueness and regularity results are well known (see *e.g.* Serrin [105]). We will fix our attention only on the behavior of the support of the solution. A well known fact is that when (2.3.12) is linear, *i.e.* with $p = 2$ and $\sigma = 1$, the solution u of (2.3.12)-(2.3.13) corresponding to data, say $f \geq 0$ and $h \geq 0$, is such that $u > 0$ in Ω. This is a trivial consequence of the *Strong Maximum Principle*[3] and can also be obtained by many others arguments, for instance the *Harnack inequality*[4] (see *e.g.* Díaz [34], and Gilbarg and Trudinger [55] for these results). When (2.3.12) is nonlinear, entirely different behavior may appear. Roughly speaking, the effective power of the diffusion term div $(|\nabla u|^{p-2}\nabla u)$ and of the absorption term $|u|^{\sigma-1}u$ vary with p and σ, generating new phenomena. Benilan et al [17] proved that if $p = 2$ in (2.3.12), Ω is an unbounded open domain and f and h have compact support, then the support of the solution contains the whole domain Ω if $\sigma \geq 1$, and if $\sigma < 1$, the solution u has compact support and so $u = 0$ in an unbounded region of Ω. Latter this result was generalized by

[2] Let us consider equation (1.1.4). Uniform ellipticity means that the ratio of maximum to minimum eigenvalue of the coefficient matrix $\mathbf{a}$ is bounded.

[3] Theorem (Strong Maximum Principle) u
Ω u

[4] Theorem (Harnack Inequality) $u \geq 0$ $f = 0$ Ω $\sup_{\Omega'} u \leq C \inf_{\Omega'} u$ $\Omega' \subset\subset \Omega$
C

Díaz and Herrero [37] for a general p obtaining the same conclusions if $\sigma \geq p-1$ and $\sigma < p-1$, respectively. The essential tool to obtain these results was the Comparison Principle.

Similar results were obtained for general second order quasi-linear equations with no monotonicity assumptions on the nonlinearities by using the energy method for free boundary problems presented in Section 1.3 (see Antontsev [5], Díaz and Véron [39]). Using a variant of this energy method, Bernis [18, 19] proved that the weak solutions of the associated Dirichlet boundary value problem to the following nonlinear higher-order partial differential equation

$$(-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha \left(|\mathrm{D}^\alpha u|^{p-1} \mathrm{sgn} \mathrm{D}^\alpha u\right) + |u|^{\sigma-1} \mathrm{sgn}\, u = f, \quad p \geq 1, \quad \sigma > 1,$$

considered in an unbounded open domain Ω with a compact boundary $\partial\Omega$, have compact support for any dimension and any $m \geq 1$ if $1 < \sigma < p$ and the support of f is bounded. Notice that for $p = 2$

$$(-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha \left(|\mathrm{D}^\alpha u|^{p-1} \mathrm{sgn} \mathrm{D}^\alpha u\right) = (-1)^m \triangle^m u,$$

and for $m = 1$

$$(-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha \left(|\mathrm{D}^\alpha u|^{p-1} \mathrm{sgn} \mathrm{D}^\alpha u\right) = -\mathrm{div}\, \left(|\nabla u|^{p-2} \nabla u\right)$$

and we fall in the case of equation (2.3.12). Bernis [20] also proved the weak solutions of the nonlinear higher-order parabolic equation

$$\frac{\partial(|u|^{q-1} \mathrm{sgn}\, u)}{\partial t} + (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha \left(|\mathrm{D}^\alpha u|^{p-1} \mathrm{sgn} \mathrm{D}^\alpha u\right) + |u|^{\sigma-1} \mathrm{sgn}\, u = f,$$

considered in $\Omega \times (0, \infty)$, where Ω is an arbitrary open domain of $\mathbb{R}^N$, with zero Dirichlet boundary conditions on $\partial\Omega \times (0, \infty)$ and $u = u_0$ in Ω for $t = 0$, have finite speed of propagation if $1 < \sigma < p$, $1 < q < p$, and the support of u_0 and the projection in Ω of the support of f are bounded.

In Antontsev et al [12, §4.7.4] was considered a flow of a nonhomogeneous non-Newtonian fluid in $Q = \Omega \times (0, T)$, Ω a subdomain of $\mathbb{R}^N$, $N \geq 1$, $0 < T \leq \infty$, governed by equations (1.4.11), (1.4.22) and (1.4.12) (for an arbitrary $N \geq 2$), with given initial conditions and zero boundary conditions. The stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = -p\mathbf{I} + \mathbf{F}(\mathbf{D}, \rho, \theta), \quad \mathbf{F}(\mathbf{0}, \rho, \theta) = \mathbf{0},$$

where $\mathbf{D}$ is the rate of strain tensor (1.4.13) and the tensor $\mathbf{F}$ is assumed to satisfy

$$\alpha |\mathbf{D}|^q \leq \mathbf{F}(\mathbf{D}) : \mathbf{D}, \quad q \in \left(\frac{2N}{N+2}, N\right), \quad \alpha = const. > 0.$$

There, it was proved that under a suitable forces field satisfying

$$-\mathbf{f}(\mathbf{x}, t, \mathbf{r}) \cdot \mathbf{r} \geq \delta |\mathbf{r}|^{1+\sigma}, \tag{2.3.14}$$

for all $\mathbf{r} \in \mathbb{R}^N$, for some $\sigma \in (0,1)$ and $\delta > 0$, the weak solutions of this problem have finite speed of propagation. In a sense, the consideration of the forces field satisfying (2.3.14) corresponds to introduce in the equation of motion (1.4.12) the absorption term $(|u_1|^{\sigma-1}u_1, \ldots, |u_N|^{\sigma-1}u_N)$. This result is still valid for the classical Navier-Stokes equations for incompressible homogeneous fluids.

Then, we collected all this knowledge and started to prove the localization effect for the weak solutions of problem (2.1.1)-(2.2.5). Using the energy method for free boundary problems presented in Section 1.3, we formally multiply (2.1.1) by a weak solution, which for now we assume its existence, and integrate over a variable subdomain of Ω. Doing this we see that, wether we integrate over a ball or a half-plane, we obtain an integral term involving the pressure p and we are not able to handle this term. Thus, we are lead firstly to eliminate the pressure of the equation (2.1.1). In this way, introducing the stream function,

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in } \Omega, \tag{2.3.15}$$

we reduce the study of problem (2.1.1)-(2.2.5) to the consideration of the following fourth-order problem, where the pressure term does not appear anymore

$$\nu \triangle^2 \psi + \tfrac{\partial f_1}{\partial y} - \tfrac{\partial f_2}{\partial x} = 0 \quad \text{in } \Omega, \tag{2.3.16}$$

$$\psi(x,0) = \psi(x,L) = \tfrac{\partial \psi}{\partial n}(x,0) = \tfrac{\partial \psi}{\partial n}(x,L) = 0 \quad \text{for } x \in (0,\infty), \tag{2.3.17}$$

$$\psi(0,y) = \textstyle\int_0^y u_*(s)ds, \ \tfrac{\partial \psi}{\partial n}(0,y) = v_*(y) \quad \text{for } y \in (0,L), \tag{2.3.18}$$

$$\psi(x,y), \ |\nabla \psi(x,y)| \to 0 \text{ as } x \to \infty \quad \text{and for } y \in (0,L). \tag{2.3.19}$$

Here $\mathbf{f} = (f_1, f_2) = (f_1(\mathbf{x}, \psi_y, -\psi_x), f_2(\mathbf{x}, \psi_y, -\psi_x))$. Then we use the energy method in the problem (2.3.16)-(2.3.19) to establish the localization effect. According to what have been said about this method in Section 1.3, we will integrate over a variable half-plane $\Omega_a = \{\mathbf{x} = (x,y) \in \Omega : x > a\}$, where a is a positive variable parameter. In order to cancel the possible non-zero boundary terms on $x = 0$, we consider the weight $\rho(\mathbf{x}) = (x-a)_+^m$, where $m \geq 2$ is an integer. Then, we formally multiply (2.3.16) by $\rho(\mathbf{x})\psi$, where ψ is a weak solution of (2.3.16)-(2.3.19), and we integrate by parts over Ω, using (2.3.17)-(2.3.19), to obtain

$$\begin{aligned} \int_\Omega (\triangle \psi)^2 (x-a)_+^m d\mathbf{x} - \int_\Omega f_1 \psi_y (x-a)_+^m d\mathbf{x} = \\ \int_\Omega f_2 \left(m\psi(x-a)_+^{m-1} + \psi_x (x-a)_+^m \right) d\mathbf{x} \\ -\nu m \int_\Omega \triangle \psi \left(2\psi_x (x-a)_+^{m-1} + (m-1)\psi(x-a)_+^{m-2} \right) d\mathbf{x}. \end{aligned} \tag{2.3.20}$$

Equality (2.3.20) and condition (2.3.14) gave us the idea for the choice of a forces field $\mathbf{f} = (f_1, f_2)$ such that

$$-f_1(\mathbf{x}, \mathbf{u})u \geq \delta\, |u|^{1+\sigma} \quad \text{for some } \delta > 0 \text{ and } \sigma \in (0,1)$$

and

$$\text{supp}\, f_2 \cap \Omega_{x_g} = \emptyset^{\,5} \quad \text{for some } x_g \in [0,\infty), \quad \Omega_{x_g} = (x_g,\infty)\times(0,L).$$

Notice that, from (2.3.20), we do not need $f_2 = 0$ in all of Ω, we may allow $f_2 \neq 0$ until a finite distance from $x = 0$ (see Figure 2.2). Therefore (2.2.9) and (2.2.10) make sense and the parameter a of the weight function ρ is chosen such that $a \geq x_g$, where x_g is given in (2.2.10).

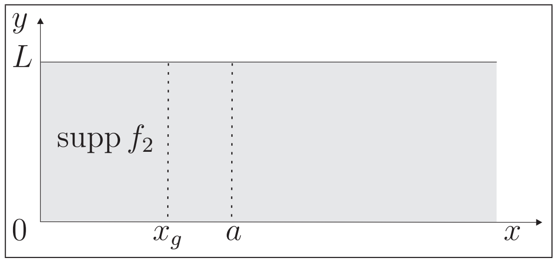

Figure 2.2: Dissipative forces field.

2.4 Weak Formulation

The presence of nonlinear terms defined by $\mathbf{f}$, and to the best of our knowledge, is new in the Fluid Mechanics setting. Thus, we collect in this section some results about existence and uniqueness. For this reason, it is useful to give a more explicit form of the forces field $\mathbf{f}$. During this section we shall assume that $\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, with $\mathbf{f}(\mathbf{x},\mathbf{u}) = (f_1(\mathbf{x},\mathbf{u}), f_2(\mathbf{x},\mathbf{u}))$, $\mathbf{u} = (u,v)$,

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = -\delta\left(|u|^{\sigma-1}u, 0\right) - \mathbf{h}(\mathbf{x},\mathbf{u}) \tag{2.4.21}$$

for some $\delta > 0$ and $0 < \sigma < 1$. Here, $\mathbf{h}(\mathbf{x},\mathbf{u})$ is a *Carathéodory function*[6] such that

$$\mathbf{h}(\mathbf{x},\mathbf{u})\cdot\mathbf{u} \geq -g(\mathbf{x}) \tag{2.4.22}$$

for every $\mathbf{u} \in \mathbb{R}^2$ and almost all $\mathbf{x} \in \Omega$ and for some function g satisfying (2.2.10). Of course the forces field $\mathbf{f}$ defined by (2.4.21) and (2.4.22) satisfies (2.2.9) and (2.2.10). We shall search for solutions such that, in addition to (2.2.5), satisfy

$$\int_\Omega |\nabla\mathbf{u}|^2 d\mathbf{x} < \infty. \tag{2.4.23}$$

Moreover, due to the fact that Poincaré's inequality (1.1.3) holds, our searched solution will be an element of the Sobolev space $\mathbf{H}^1(\Omega)$ simplifying, in this way, the functional framework needed for other unbounded domains. To define the notion of weak solution, we introduce the functional spaces

$$\widetilde{\mathbf{H}}(\Omega) = \left\{\mathbf{u}\in\mathbf{H}(\Omega) : \mathbf{u}(0,.) = \mathbf{u}_*(.),\ \mathbf{u}(.,0) = \mathbf{u}(.,L) = \mathbf{0},\ \lim_{x\to\infty}|\mathbf{u}| = 0\right\},$$

[5]It is implied we are talking about the projection in Ω of the support of f_2.

[6] $\mathbf{h}(\mathbf{x},\mathbf{u})$ is measurable in $\mathbf{x}$ for every $\mathbf{u}\in\mathbb{R}^2$ and continuous in $\mathbf{u}$ for almost all $\mathbf{x}\in\Omega$.

$$\widetilde{\mathbf{H}}_0(\Omega) = \left\{ \mathbf{u} \in \mathbf{H}(\Omega) : \mathbf{u}(0,.) = \mathbf{u}(.,0) = \mathbf{u}(.,L) = \mathbf{0}, \ \lim_{x\to\infty} |\mathbf{u}| = 0 \right\},$$

where $\mathbf{H}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathrm{div}\mathbf{u} = 0\}$.

Definition 2.4.1 *A vector function* $\mathbf{u}$ *is a weak solution of the Stokes problem (2.1.1)-(2.2.5), with the forces field* $\mathbf{f}$ *satisfying (2.4.21) and (2.4.22) if:*
(i) $\mathbf{u} \in \widetilde{\mathbf{H}}(\Omega)$, $\mathbf{f}(\mathbf{x},\mathbf{u}) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(ii) for every $\varphi \in \widetilde{\mathbf{H}}_0(\Omega) \cap \mathbf{L}^\infty(\Omega)$ *with compact support,*

$$\nu \int_\Omega \nabla \mathbf{u} : \nabla \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x}.$$

Under a growth condition on the function $\mathbf{h}$ we are able to prove an existence theorem.

Theorem 2.4.1 *Let us assume* $\mathbf{u}_* \in \mathbf{H}^{\frac{1}{2}}(0,L)$, $\mathbf{f}(\mathbf{x},\mathbf{u})$ *satisfies (2.4.21) and (2.4.22) and the following growth condition holds: there exist some positive constant* C*, a function* $G \in \mathrm{L}^p(\Omega)$*, for some* $p > 1$ *and* $s \in (0,1)$*, such that*

$$|\mathbf{h}(\mathbf{x},\mathbf{u})| \le C\,|\mathbf{u}|^s + G(\mathbf{x}), \tag{2.4.24}$$

for every $\mathbf{u} \in \mathbb{R}^2$ *and almost everywhere in* Ω*. Then, there exists, at least, one weak solution* $\mathbf{u}$ *of problem (2.1.1)-(2.2.5). Moreover,* $\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u}$ *lies in* $\mathrm{L}^1(\Omega)$ *and* $\mathbf{u}$ *satisfies to the energy estimate*

$$\int_\Omega \left(|\nabla \mathbf{u}|^2 + |u|^{1+\sigma} \right) d\mathbf{x} \le C, \tag{2.4.25}$$

where $C = C\left(L, \delta, s, p, \nu, \sigma, \|g\|_{\mathrm{L}^1(\Omega^{x_g})}, \|G\|_{\mathrm{L}^p(\Omega)}, \|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right)$.

Proof: We split the proof into four steps.
First step. We start by considering, for a given $N \in \mathbb{N}$, the auxiliary problem in $\Omega^N = (0,N) \times (0,L)$

$$-\nu \triangle \mathbf{u}^N = \mathbf{f}(\mathbf{x},\mathbf{u}^N) - \nabla p^N \quad \text{in } \Omega^N, \tag{2.4.26}$$

$$\mathrm{div}\mathbf{u}^N = 0 \quad \text{in } \Omega^N, \tag{2.4.27}$$

$$\mathbf{u}^N = \mathbf{u}_*(y) \text{ for } x = 0 \tag{2.4.28}$$

$$\mathbf{u}^N = \mathbf{0} \text{ for } x = N \text{ and } y = 0,\ L. \tag{2.4.29}$$

With no lost of generality, we assume $N > 1$, and let $\mathbf{U}^1$ be an extension of $\mathbf{u}_*$ to $\Omega^1 = (0,1) \times (0,L)$ such that:

$$\begin{array}{l} \text{(i) } \mathbf{U}^1 \in \mathbf{H}^1(\Omega^1); \\ \text{(ii) } \mathrm{div}\mathbf{U}^1 = 0 \ \text{ in } \ \Omega^1; \\ \text{(iii) } \mathbf{U}^1 = \mathbf{u}_* \text{ on } x = 0, \quad \mathbf{U}^1 = \mathbf{0} \text{ on } x = 1 \text{ and on } y = 0,\ L. \end{array} \tag{2.4.30}$$

Such an extension[7] exists (see *e.g.* Galdi [49, §III.3]) and, moreover,

$$\|\mathbf{U}^1\|_{\mathbf{H}^1(\Omega^1)} \le C \|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}, \quad C = C(L). \tag{2.4.31}$$

[7]Where (iii) is implied in the trace sense.

Now, we consider the extension $\mathbf{U}^N$ to Ω^N such that $\mathbf{U}^N = \mathbf{U}^1$ if $x < 1$ and $\mathbf{U}^N = \mathbf{0}$ if $x \geq 1$. From what we said above, $\mathbf{U}^N \in \mathbf{H}^1(\Omega^N)$ and we have

$$\|\mathbf{U}^N\|_{\mathbf{H}^1(\Omega^N)} \leq C\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}, \quad C = C(L). \tag{2.4.32}$$

Second step. We look for solutions $\mathbf{u}^N$ of the form $\mathbf{u}^N = \mathbf{w}^N + \mathbf{U}^N$, where $\mathbf{U}^N$ is the extension given in the *First step* and $\mathbf{w}^N$ solves the following problem in Ω^N

$$-\nu\triangle\mathbf{w}^N = \mathbf{g}(\mathbf{x}, \mathbf{w}^N) - \nabla p^N \quad \text{in } \Omega^N, \tag{2.4.33}$$

$$\text{div}\mathbf{w}^N = 0 \quad \text{in } \Omega^N, \tag{2.4.34}$$

$$\mathbf{w}^N = \mathbf{0} \quad \text{for } x = 0,\ N \text{ and } y = 0,\ L, \tag{2.4.35}$$

where

$$\mathbf{g}(\mathbf{x}, \mathbf{w}^N) = \mathbf{f}(\mathbf{x}, \mathbf{w}^N + \mathbf{U}^N) + \nu\triangle\mathbf{U}^N.$$

From (2.4.24),

$$\left|\mathbf{f}(\mathbf{x}, \mathbf{v^N})\right| \leq a(\mathbf{x})|\mathbf{v}^N|^\sigma + C\left|\mathbf{v}^N\right|^s + |G(\mathbf{x})| \tag{2.4.36}$$

for every $\mathbf{v}^N \in \mathbb{R}^2$ and almost all $\mathbf{x} \in \Omega^N$, where $a \geq 0$ is a measurable function in Ω^N with

$$a \in \mathrm{L}^{\frac{2q}{2-q\sigma}}(\Omega^N).$$

From (2.4.36) and after made use of Young's inequality with ε,

$$|\mathbf{f}(\mathbf{x}, \mathbf{v}^N)|^q \leq C_1\left(\varepsilon|\mathbf{v}^N|^2 + C_2\, a^{\frac{2q}{2-q\sigma}} + C\left|\mathbf{v}^N\right|^{qs} + |G(\mathbf{x})|^q\right), \tag{2.4.37}$$

where $C_1 = C_1(q)$, $C_2 = C_2(q, \varepsilon, \sigma)$ are positive constants and C is given in (2.4.24). Then, given $\mathbf{v}^N \in \mathbf{L}^2(\Omega^N)$, we see that from (2.4.37) and by applying Hölder's inequality $\mathbf{f}(\mathbf{x}, \mathbf{v}^N) \in \mathbf{L}^q(\Omega^N)$, with $q = \min(2/\sigma, 2/s, p)$. In consequence, there exists a unique weak solution $\mathbf{w}^N \in \mathbf{H}_0^1(\Omega^N)$ of the problem (2.4.33)-(2.4.35) with the body forces given by $\mathbf{g}(\mathbf{x}, \mathbf{v}^N)$ (see *e.g.* Galdi [49, §IV.6]). Thus, we can define a nonlinear operator $\Lambda : \mathbf{L}^2(\Omega^N) \times [0, 1] \to \mathbf{L}^2(\Omega^N)$, by setting

$$\Lambda(\mathbf{v}^N, \lambda) = \mathbf{w}^N, \tag{2.4.38}$$

associated to the problem (2.4.33)-(2.4.35) with the body forces field given by $\lambda\mathbf{g}(\mathbf{x}, \mathbf{v}^N)$. Multiplying (2.4.33), with $\lambda\mathbf{g}(\mathbf{x}, \mathbf{w}^N)$, by $\mathbf{w}^N$, integrating by parts over Ω^N, using (2.4.34)-(2.4.35) and the Sobolev imbedding

$$\mathbf{H}^1(\Omega^N) \to \mathbf{L}^q(\Omega^N), \quad 1 \leq q < \infty, \tag{2.4.39}$$

we obtain the *a priori* estimate

$$\|\mathbf{w}^N\|_{\mathbf{H}_0^1(\Omega^N)} < C, \tag{2.4.40}$$

where $C = C\,(L, p, s, \nu, R)$ and R is taken such that

$$R > \max\left(\|G\|_{\mathrm{L}^p(\Omega)}, \|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}, 1\right).$$

Then, from (2.4.40), the operator (2.4.38) maps a ball in $\mathbf{L}^2(\Omega^N)\times[0,1]$ into a bounded subset of $\mathbf{H}_0^1(\Omega^N)$ and from the Sobolev compact imbedding $\mathbf{H}_0^1(\Omega^N)\to\mathbf{L}^2(\Omega^N)$, it is a completely continuous operator. Moreover $\Lambda(\mathbf{v}^N,0)=\mathbf{0}$ and from the Leray-Schauder fixed point theorem, $\Lambda(\cdot,1)$ has a fixed point, $\Lambda(\mathbf{w}^N,1)=\mathbf{w}^N$. This proves the existence of, at least, one weak solution $\mathbf{w}^N\in\mathbf{H}_0^1(\Omega^N)$ of the problem (2.4.33)-(2.4.35), with $\mathbf{g}(\mathbf{x},\mathbf{w}^N)$. Consequently the existence of, at least, one weak solution $\mathbf{u}^N\in\mathbf{H}^1(\Omega^N)$ of the problem (2.4.26)-(2.4.29) is assured.

Third step. Multiplying formally (2.4.33), with $\mathbf{g}(\mathbf{x},\mathbf{w}^N)$, by $\mathbf{w}^N$, integrating by parts over Ω^N, using (2.4.21) and (2.4.22), (2.4.34)-(2.4.35), the Sobolev imbedding (2.4.39), Young's inequality with a suitable ε and finally replacing $\mathbf{w}^N=\mathbf{u}^N-\mathbf{U}^N$, we obtain the following estimate independent of N

$$\int_{\Omega^N}\left(|\nabla\mathbf{u}^N|^2+|u^N|^{1+\sigma}\right)\,d\mathbf{x}\le C, \tag{2.4.41}$$

with $C=C\left(L,\delta,s,p,\nu,\|g\|_{\mathrm{L}^1(\Omega^{x_g})},\|G\|_{\mathrm{L}^p(\Omega)},\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right)$.

Fourth step. Now, for each $N\in\mathbb{N}$, we consider a sequence $\mathbf{u}_k^N$ of weak solutions to problems (2.4.26)-(2.4.29) and thus satisfying (2.4.41). Then, because of the reflexivity of the space $\mathbf{H}^1(\Omega^N)$, there exists a subsequence which we still denote by $\mathbf{u}_k^N$, such that

$$\mathbf{u}_k^N\to\mathbf{u}^N \quad \text{weakly in } \mathbf{H}^1(\Omega^N), \quad \text{as } k\to\infty,$$

and because $\mathbf{h}$ is a Carathéodory function,

$$\mathbf{h}\left(\mathbf{x},\mathbf{u}_k^N\right)\to\mathbf{h}\left(\mathbf{x},\mathbf{u}^N\right) \quad \text{in } \mathbf{L}^1(\Omega^N), \quad \text{as } k\to\infty.$$

In consequence, using a standard diagonal process and that $\mathbf{h}(\mathbf{x},\mathbf{u}_k^N)$ is a Carathéodory function, we can choose a subsequence $\mathbf{u}_k^{N_k}$ such that, for every $R>0$,

$$\mathbf{u}_k^{N_k}\to\mathbf{u} \quad \text{weakly in } \mathbf{H}^1(\Omega^R), \quad \text{as } k\to\infty,$$

and

$$\mathbf{h}\left(\mathbf{x},\mathbf{u}_k^{N_k}\right)\to\mathbf{h}\left(\mathbf{x},\mathbf{u}\right) \quad \text{in } \mathbf{L}^1(\Omega^R), \quad \text{as } k\to\infty.$$

In addition, $\mathbf{u}$ satisfies to the energy estimate (2.4.25). Now, by the Sobolev imbedding (2.4.39), we get that $\mathbf{f}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}\in\mathrm{L}^1(\Omega)$, $\mathbf{h}(\mathbf{x},\mathbf{u})\in\mathbf{L}^1(\Omega)$ and $\mathbf{u}$ is a weak solution to the nonlinear problem (2.1.1)-(2.2.5).□

In some situations, we can prove the existence result by dropping the growth condition (2.4.24). But then, we need to assume an extra condition on the field $\mathbf{h}$

$$H_K\in\mathrm{L}^1(\Omega), \quad \text{for all } K>0, \quad H_K(\mathbf{x})=\sup_{|\mathbf{u}|\le K}|\mathbf{h}(\mathbf{x},\mathbf{u})|. \tag{2.4.42}$$

Moreover, in order to control the convergence of suitable approximations, we need to assume the vectors angle condition: there exists $\varepsilon>0$ such that

$$|\angle(\mathbf{h}(\mathbf{x},\mathbf{u}),\mathbf{u})|\notin\left(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon\right) \tag{2.4.43}$$

for every $|\mathbf{u}|>K$ and almost everywhere in Ω.

Remark 2.4.1 *Notice that (2.4.21), (2.4.22), (2.4.42) and (2.4.43) do not imply any upper restriction on the growth of* $|\mathbf{f}(\mathbf{x},\mathbf{u})|$ *with respect to* $\mathbf{u}$ *and due to that, sometimes, this type of terms are called strongly nonlinear.*

Theorem 2.4.2 *Theorem 2.4.1 is still valid if we assume the forces field* $\mathbf{f}$ *satisfies (2.4.21), (2.4.22), (2.4.42) and (2.4.43). Here the energy estimate (2.4.25) takes the form*

$$\int_{\Omega}\left(|\nabla\mathbf{u}|^2+|u|^{1+\sigma}+|\mathbf{h}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}|\right)\,d\mathbf{x}\leq C, \tag{2.4.44}$$

where, now, $C=C\left(L,\delta,\nu,\sigma,\|g\|_{\mathrm{L}^1(\Omega^{x_g})},\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right)$.

Proof: Here we split the proof into five steps.
First step. All that is written in the *First step* of the *Proof* of Theorem 2.4.1 is valid here. Moreover, using Hölder's inequality, one can prove

$$\int_{\Omega^N}|\mathbf{U}^N|^p d\mathbf{x}\leq C\|\mathbf{u}_*\|^p_{\mathbf{H}^{\frac{1}{2}}(0,L)}\quad\text{for}\ \ 1\leq p<2,\quad C=C(L,p). \tag{2.4.45}$$

Second step. Firstly, we consider the intermediary case in which we assume, additionally,

$$|\mathbf{h}(\mathbf{x},\mathbf{u})|\leq C, \tag{2.4.46}$$

for some positive constant C, for every $\mathbf{u}\in\mathbb{R}^2$ and almost all $\mathbf{x}\in\Omega$.

We look for solutions $\mathbf{u}^N$ of the form $\mathbf{u}^N=\mathbf{w}^N+\mathbf{U}^N$, where $\mathbf{U}^N$ is the extension given in the *First step* of the *Proof* of Theorem 2.4.1 and $\mathbf{w}^N$ solves the problem (2.4.33)-(2.4.35).

Notice that, in the special case of (2.4.46), $\mathbf{f}(\mathbf{x},\mathbf{v}^N)$ satisfies

$$|\mathbf{f}(\mathbf{x},\mathbf{v}^N)|\leq a(\mathbf{x})|\mathbf{v}^N|^{\sigma}+b(\mathbf{x}), \tag{2.4.47}$$

for every $\mathbf{v}^N\in\mathbb{R}^2$ and almost all $\mathbf{x}\in\Omega^N$, where $a,b\geq 0$ are measurable functions in Ω^N with

$$a\in\mathrm{L}^{\frac{2}{1-\sigma}}(\Omega^N),\quad b\in\mathrm{L}^2(\Omega^N). \tag{2.4.48}$$

From (2.4.47) and after made use of Young's inequality with ε,

$$|\mathbf{f}(\mathbf{x},\mathbf{v}^N)|^2\leq 4\left(\varepsilon|\mathbf{v}^N|^2+C\,a^{\frac{2}{1-\sigma}}+b^2\right),\quad C=C(\varepsilon,\sigma). \tag{2.4.49}$$

Then, given $\mathbf{v}^N\in\mathbf{L}^2(\Omega^N)$, $\mathbf{f}(\mathbf{x},\mathbf{v}^N)\in\mathbf{L}^2(\Omega^N)$ and there exists a unique weak solution $\mathbf{w}^N\in\mathbf{H}^1_0(\Omega^N)$ of the problem (2.4.33)-(2.4.35) with the body forces given by $\mathbf{g}(\mathbf{x},\mathbf{v}^N)$ (see *e.g.* Galdi [49, §IV.6]). Thus, we can define a nonlinear operator $\Lambda:\mathbf{L}^2(\Omega^N)\times[0,1]\to\mathbf{L}^2(\Omega^N)$, by setting (2.4.38), associated to the problem (2.4.33)-(2.4.35) with the body forces field given by $\lambda\mathbf{g}(\mathbf{x},\mathbf{v}^N)$. Multiplying (2.4.33), with $\lambda\mathbf{g}(\mathbf{x},\mathbf{w}^N)$, by $\mathbf{w}^N$, integrating by parts over Ω^N, using (2.4.34)-(2.4.35), Sobolev imbedding (2.4.39) with $q=2$ and (2.4.49) with ε chosen in a reasonable way, we obtain the *a priori* estimate (2.4.40), where now

$$C=C\left(L,\nu,R,\|a\|_{\mathrm{L}^{\frac{2}{1-\sigma}}(\Omega^N)},\|b\|_{\mathrm{L}^2(\Omega^N)}\right)$$

and R is taken such that
$$R > \max\left(\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}, 1\right).$$
Then, from (2.4.40), the operator (2.4.38) maps a ball in $\mathbf{L}^2(\Omega^N)\times[0,1]$ into a bounded subset of $\mathbf{H}_0^1(\Omega^N)$ and from the Sobolev compact imbedding $\mathbf{H}_0^1(\Omega^N) \to \mathbf{L}^2(\Omega^N)$, it is a completely continuous operator. Then the existence of a weak solution $\mathbf{u}^N \in \mathbf{H}^1(\Omega^N)$ of the problem (2.4.26)-(2.4.29) follows as in the *Second step* of the *Proof* of Theorem 2.4.1.

Third step. We shall prove the *a priori* estimate (independent of N) for $\mathbf{u}^N$
$$\int_{\Omega^N}\left(|\nabla\mathbf{u}^N|^2 + |u^N|^{1+\sigma} + \left|\mathbf{h}(\mathbf{x},\mathbf{u}^N)\cdot\mathbf{u}^N\right|\right)\,d\mathbf{x} \le C \tag{2.4.50}$$
where $C = C\left(L,\delta,\nu,\sigma,\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}, \|g\|_{\mathrm{L}^1(\Omega^{xg})}\right)$. We point out that from assumptions (2.2.10) and (2.4.22),
$$|\mathbf{h}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}| \le \mathbf{h}(\mathbf{x},\mathbf{u})\cdot\mathbf{u} + 2g(\mathbf{x}), \tag{2.4.51}$$
for every $\mathbf{u}\in\mathbb{R}^2$ and almost all $\mathbf{x}\in\Omega$.

In the following energy relation satisfied by $\mathbf{u}^N$
$$\nu\int_{\Omega^N}\nabla\mathbf{u}^N : (\nabla\mathbf{u}^N - \nabla\mathbf{U}^N)d\mathbf{x} = \int_{\Omega^N}\mathbf{f}\cdot(\mathbf{u}^N - \mathbf{U}^N)d\mathbf{x},$$
we use assumption (2.4.21), next we add $\left|\mathbf{h}(\mathbf{x},\mathbf{u}^N)\cdot\mathbf{u}^N\right|$ to both sides of the resultant equation, we use assumptions (2.2.10), (2.4.46) and also (2.4.51), then we apply Young's inequality with a suitable $\varepsilon > 0$ and we obtain
$$\int_{\Omega^N}\left(|\nabla\mathbf{u}^N|^2 + |u^N|^{1+\sigma} + \left|\mathbf{h}(\mathbf{x},\mathbf{u}^N)\cdot\mathbf{u}^N\right|\right)\,d\mathbf{x} \le$$
$$C\left(\int_{\Omega^N}\left(|\mathbf{U}^N| + |\mathbf{U}^N|^{1+\sigma}\right)\,d\mathbf{x} + \int_{\Omega^N}|\nabla\mathbf{U}^N|^2 d\mathbf{x} + \|g\|_{\mathrm{L}^1(\Omega^{xg})}\right),$$
where $C = C(\delta,\nu,\sigma)$. Then, we use (2.4.32) and (2.4.45), and we get
$$\int_{\Omega^N}\left(|\nabla\mathbf{u}^N|^2 + |u^N|^{1+\sigma} + \left|\mathbf{h}(\mathbf{x},\mathbf{u}^N)\cdot\mathbf{u}^N\right|\right)\,d\mathbf{x} \le$$
$$C\left(\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)} + \|\mathbf{u}_*\|^{1+\sigma}_{\mathbf{H}^{\frac{1}{2}}(0,L)} + \|\mathbf{u}_*\|^{2}_{\mathbf{H}^{\frac{1}{2}}(0,L)} + \|g\|_{\mathrm{L}^1(\Omega^{xg})}\right),$$
where $C = C(L,\delta,\nu,\sigma)$. Finally, once that $0 < \sigma < 1$, we can use the algebraic inequality asserting that $A + A^{1+\sigma} + A^2 \le C(A^2+1)$, for every $A \ge 0$ and C a positive constant, to obtain (2.4.50).

Fourth step. Now, for each $N\in\mathbb{N}$, we consider a sequence $\mathbf{u}_k^N$ of weak solutions to problems (2.4.26)-(2.4.29) and thus satisfying (2.4.50). In consequence, using a standard diagonal process and that $\mathbf{h}(\mathbf{x}, \mathbf{u}_k^N)$ is a Carathéodory function, we can choose a subsequence $\mathbf{u}_k^{N_k}$, such that

$$\mathbf{u}_k^{N_k} \to \mathbf{u} \ \text{ weakly in } \mathbf{H}^1(\Omega^R), \quad \text{as} \quad k \to \infty,$$

$$\mathbf{h}\left(\mathbf{x}, \mathbf{u}_k^{N_k}\right) \to \mathbf{h}\left(\mathbf{x}, \mathbf{u}\right) \ \text{ in } \mathbf{L}^1(\Omega^R), \quad \text{as} \quad k \to \infty,$$

for every $R > 0$ and $\mathbf{u}$ is a weak solution to the problem (2.1.1)-(2.2.5), once we assume condition (2.4.46). In addition, $\mathbf{u}$ satisfies to the energy estimate (2.4.44).

Fifth step. We proceed with a truncation and approximation argument to deal with the general case, *i.e.* the case without condition (2.4.46). We adapt the arguments of Brézis and Browder [27] (see also Bernis [22]). We consider the n-radial truncation $\mathbf{h}_n(\mathbf{x}, \mathbf{u})$ defined by

$$\mathbf{h}_n(\mathbf{x}, \mathbf{u}) = \begin{cases} \mathbf{h}(\mathbf{x}, \mathbf{u}) & \text{if } |\mathbf{u}| \le n, \\ \mathbf{h}(\mathbf{x}, (n\cos\theta, n\sin\theta)) & \text{if } \mathbf{u} = (r\cos\theta, r\sin\theta) \text{ and } r > n. \end{cases} \tag{2.4.52}$$

It was shown in Vrabie [120, Lemma 3.4.3] that $\mathbf{h}_n(\mathbf{x}, \mathbf{u})$ is continuous and bounded on $\mathbf{u}$ for almost all $\mathbf{x} \in \Omega$. Moreover, $\mathbf{h}_n(\mathbf{x}, \mathbf{u})$ satisfies assumptions (2.4.21), (2.4.22), (2.4.42), (2.4.43) and (2.4.46) with the same functions g and H_K. From Step 4, it follows that there exists a weak solution $\mathbf{u}_n$ of problem (2.1.1)-(2.2.5), with $\mathbf{f}(\mathbf{x}, \mathbf{u_n})$, and which satisfies (2.4.44) with the constant C independent of n. Therefore, there exists a subsequence, which we still denote by $\mathbf{u}_n$, such that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } \mathbf{H}^1(\Omega), \quad \text{as} \quad n \to \infty. \tag{2.4.53}$$

Then

$$\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n) \to \mathbf{h}(\mathbf{x}, \mathbf{u}) \ \text{ for almost all } \ \mathbf{x} \in \Omega, \quad \text{as} \quad n \to \infty. \tag{2.4.54}$$

We have, from (2.4.44),

$$\int_\Omega |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n) \cdot \mathbf{u}_n| \, d\mathbf{x} \le C,$$

with C independent of n. Therefore, from (2.4.53), (2.4.54) and Fatou's Lemma, $\mathbf{h}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \in \mathrm{L}^1(\Omega)$ and by the Sobolev imbedding (2.4.39) with $q = 1$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \in \mathrm{L}^1(\Omega)$. On one hand we have, for any $K > 0$,

$$|\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n)| \le |\mathbf{h}(\mathbf{x}, \mathbf{u}_n)| \le H_K(\mathbf{x}), \quad \text{if} \quad |\mathbf{u}_n| < K.$$

On the other, if $|\mathbf{u}_n| \ge K$, then

$$K \int_\Omega |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n)| \, |\cos\beta_n| \, d\mathbf{x} \le \int_\Omega |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n) \cdot \mathbf{u}_n| \, d\mathbf{x} \le C,$$

where $\beta_n(\mathbf{x}) = \angle(\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n(\mathbf{x})), \mathbf{u}_n(\mathbf{x}))$. Then $\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n(\mathbf{x})) \cos\beta_n(\mathbf{x})$ is uniformly integrable since

$$\int_\mathrm{E} |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n)| \, |\cos\beta_n| \, d\mathbf{x} \le \frac{C}{K} + \int_\mathrm{E} H_K(\mathbf{x}) \, d\mathbf{x},$$

for any measurable subset E of Ω. Indeed, once we assume (2.4.42), given a $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\int_{\mathrm{E}} H_K(\mathbf{x}) d\mathbf{x} < \varepsilon \quad \text{if} \quad |\mathrm{E}| < \delta$$

and taking K such that $C/K < \varepsilon$, we get that

$$\int_{\mathrm{E}} |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n)| \, |\cos \beta_n| \, d\mathbf{x} < 2\varepsilon. \tag{2.4.55}$$

Then, by Vitali's convergence theorem,

$$\mathbf{h}_n(., \mathbf{u}_n(.)) \cos \beta_n(.) \to \mathbf{h}(., \mathbf{u}(.)) \cos \beta(.) \quad \text{in} \quad \mathbf{L}^1_{\text{loc}}(\Omega).$$

From assumption (2.4.43), $|\cos \beta_n(.)|$, $|\cos \beta(.)| > \eta$ for some $\eta > 0$ and we deduce, from (2.4.55), that

$$\int_{\mathrm{E}} |\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n)| \, d\mathbf{x} < \frac{2\varepsilon}{\eta}.$$

Again, by Vitali's convergence theorem, we conclude that

$$\mathbf{h}_n(\mathbf{x}, \mathbf{u}_n) \to \mathbf{h}(\mathbf{x}, \mathbf{u}) \ \text{ in } \ \mathbf{L}^1_{\text{loc}}(\Omega).$$

Then $\mathbf{h}(\mathbf{x}, \mathbf{u}) \in \mathbf{L}^1_{\text{loc}}(\Omega)$ and $\mathbf{u}$ is a weak solution to the nonlinear problem (2.1.1)-(2.2.5).$\square$

Remark 2.4.2 *Notice that* $\mathbf{h}(\mathbf{x}, \mathbf{u})$ *is not assumed to be nondecreasing in* $\mathbf{u}$ *and* $\mathbf{h}(\mathbf{x}, \mathbf{0})$ *need not to be zero.*

Remark 2.4.3 *The importance of some kind of sign condition such as (2.4.22) is clear from the above proof. This is stressed by the following nonexistence result of Walter [121]: Assume that* $f \in \mathrm{C}(\mathbb{R})$, $f > 0$ *and* $f(s) \geq s^p$ *for some* $p > 1$ *and for all positive* s *large enough. Then there is no function* $u \in \mathrm{C}^{2m}(\mathbb{R}^2)$ *satisfying* $\triangle^m u = f(u)$ *in* $\mathbb{R}^2$.

Remark 2.4.4 *Condition (2.4.43) is necessary if we do not know any growth information on* $\mathbf{h}(\mathbf{x}, \mathbf{u})$ *as proves the following counter-example*[8] *for a Coriolis type body forces field*

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) = \exp(|\mathbf{u}|^{2+\varepsilon})(-v, u), \quad \text{for some } \ \varepsilon > 0.$$

Then, the function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ *obtained by passing to the limit in the* Fifth step *of the above proof is, in general, such that* $\mathbf{h}(\mathbf{x}, \mathbf{u}) \notin \mathbf{L}^1(\Omega)$. *Notice that for such special* $\mathbf{h}(\mathbf{x}, \mathbf{u})$, *we have that* $|\angle(\mathbf{h}(\mathbf{x}, \mathbf{u}), \mathbf{u})| = \pi/2$ *and so condition (2.4.43) fails.*

Remark 2.4.5 *The truncation (2.4.52) is good in the sense that the coerciveness*[9] *associated with the term* $\delta(|u|^{\sigma-1}u, 0)$ *is not destroyed.*

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[8] We thank to Professor A.V. Kazhikov who has alerted us for the possible occurrence of this kind of forces field.

[9] A mapping T from a Banach space X to its dual X′ is said to be if $\|u\|^{-1}\langle T(u), u\rangle_{\mathrm{X}'\times\mathrm{X}} \to +\infty$ as $\|u\| \to +\infty$.

Remark 2.4.6 *The above existence theorems admit many different variations ($\sigma \geq 1$, case of $\sigma = 0$, more general unbounded sets Ω, etc.), but they are not considered here. Our presentation is strictly motivated by the results on the localization effects.*

Moreover, if we assume the non-increasing condition

$$(\mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \leq 0 \tag{2.4.56}$$

holds for every $\mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^2$ and almost all $\mathbf{x} \in \Omega$, we are able to establish an uniqueness result. In order to prove the uniqueness result it is important to know under what conditions is the function on the left-hand side of (2.4.56) integrable in Ω (for the Lebesgue measure) and if so, does

$$\int_\Omega (\mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2)\ d\mathbf{x} =$$

$$\langle \mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)}\ ?$$

The answer to this question is given by the following auxiliary result.

Lemma 2.4.1 (Brézis and Browder [28]) *Assume that $\mathbf{T} \in \mathbf{L}^1_{\text{loc}}(\Omega) \cap \mathbf{H}^{-1}(\Omega)$ and $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ are such that $\mathbf{T}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \geq 0$ almost everywhere in Ω. Then $\mathbf{T} \cdot \mathbf{u} \in \mathrm{L}^1(\Omega)$ and*

$$\langle \mathbf{T}, \mathbf{u} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = \int_\Omega \mathbf{T}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x}.$$

Theorem 2.4.3 *Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two weak solutions of problem (2.1.1)-(2.2.5) and let us assume (2.4.56) holds. Then $\mathbf{u}_1 = \mathbf{u}_2$.*

Proof: Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two weak solutions. Then, according to Definition 2.4.1, $\mathbf{u_1} - \mathbf{u_2} \in \mathbf{H}_0^1(\Omega)$. Hence, $\mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2) \in \mathbf{H}^{-1}(\Omega)$. But, since (2.4.56) holds, we can use Lemma 2.4.1 and thus

$$\nu \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2\ d\mathbf{x} = \int_\Omega (\mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2)\ d\mathbf{x} \tag{2.4.57}$$

which implies that

$$\int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2\, d\mathbf{x} = 0$$

and from Poincaré's inequality, we get the result.□

Remark 2.4.7 *There is a special situation where the non-increasing condition (2.4.56) can be dropped. This is the case when the forces field is given by the simpler expression (2.2.11). The proof consists in considering the forces field given by (2.2.11) in (2.4.57), to obtain*

$$\nu \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2\ d\mathbf{x} + \delta \int_\Omega (|u_1|^{\sigma-1}u_1 - |u_2|^{\sigma-1}u_2, 0) \cdot (\mathbf{u}_1 - \mathbf{u}_2)\, d\mathbf{x} = 0$$

Then, applying the inequality

$$\sigma\,|\xi-\eta|^{\sigma+1} \le \left(|\xi|^{\sigma-1}\,\xi - |\eta|^{\sigma-1}\,\eta\right)(\xi-\eta)\left(|\xi|^{\sigma+1}+|\eta|^{\sigma+1}\right)^{\frac{1-\sigma}{1+\sigma}}, \quad 0<\sigma<1,$$

we can write

$$\nu\int_\Omega |\nabla(\mathbf{u}_1-\mathbf{u}_2)|^2\,d\mathbf{x} + \delta\,\sigma\int_\Omega |u_1-u_2|^{\sigma+1}\left(|u_1|^{\sigma+1}+|u_2|^{\sigma+1}\right)^{\frac{\sigma-1}{\sigma+1}}\,d\mathbf{x} \le 0$$

and from Poincaré's inequality, we get $\mathbf{u}_1 = \mathbf{u}_2$.

2.5 Stopping Effect

We notice that the existence of a weak solution having a finite global energy

$$E := \int_\Omega \left(|\nabla \mathbf{u}|^2 + |u|^{1+\sigma}\right)\,d\mathbf{x} \tag{2.5.58}$$

has been established in the previous section (see Theorems 2.4.1 and 2.4.2). In this section we establish the localization effect already arisen in Section 2.3.

Theorem 2.5.1 *Assume that* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). If* $\mathbf{u}$ *is any weak solution of the problem (2.1.1)-(2.2.5), with finite energy (2.5.58), then* $\mathbf{u}(x,y)=\mathbf{0}$ *for* $x>a'$*, where* $a'=a'(E,L,\delta,\nu,\sigma)$ *is a positive constant.*

In order to prove the localization effect, it is useful to work with the associated stream function ψ. We recall that due to the incompressibility condition (2.1.2), there exists a function ψ such that (2.3.15) holds (see *e.g.* Feistauer [45, Theorem 2.5.1]). In this way, by classical methods, applying the curl to (2.1.1) and then replacing $(u,v)=(\psi_y,-\psi_x)$ (see *e.g.* Ladyzhenskaya [72, §2.3]), we can reduce the study of problem (2.1.1)-(2.2.5), to the consideration of problem (2.3.16)-(2.3.19), where the pressure term does not appear. It is worth to notice the stream function (2.3.15) is defined up to a constant and we can fix it by choosing $\psi(0,0)=0$. Otherwise we can always redefine the stream function in order to obtain (2.3.16)-(2.3.19). The notion of weak solution for problem (2.3.16)-(2.3.19) is adapted to the information we have on function $\mathbf{f}(\mathbf{x},\psi_y,-\psi_x)$.

Definition 2.5.1 *A function* ψ *is called a weak solution of problem (2.3.16)-(2.3.19), if:*
(i) $\psi\in \mathrm{H}^2(\Omega)$, $\mathbf{f}(\mathbf{x},\psi_y,-\psi_x)\in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(ii) $\psi(0,y)=\int_0^y u_*(s)ds$, $\frac{\partial\psi}{\partial n}(0,y)=v_*(y)$ *for* $y\in(0,L)$, $\psi(x,0)=\psi(x,L)=\frac{\partial\psi}{\partial n}(x,0)=\frac{\partial\psi}{\partial n}(x,L)=0$ *for* $x\in(0,\infty)$ *and* $\psi(0,0)=\psi(0,L)=0$;
(iii) ψ, $|\nabla\psi|\to 0$ *when* $x\to\infty$;
(iv) for every $\phi\in \mathrm{H}^2_0(\Omega)\cap \mathrm{W}^{1,\infty}(\Omega)$ *with compact support,*

$$\nu\int_\Omega \triangle\psi\triangle\phi\,d\mathbf{x} - \int_\Omega \left(f_1(\mathbf{x},\psi_y,-\psi_x)\phi_y - f_2(\mathbf{x},\psi_y,-\psi_x)\phi_x\right)\,d\mathbf{x} = 0.$$

The next lemma tells us that a weak solution of problem (2.1.1)-(2.2.5) is connected with a weak solution of problem (2.3.16)-(2.3.19) by (2.3.15).

Lemma 2.5.1 *If* $\mathbf{u}$ *is a weak solution of problem (2.1.1)-(2.2.5) in the sense of Definition 2.4.1, then* ψ *given by (2.3.15), is a weak solution of problem (2.3.16)-(2.3.19) in the sense of Definition 2.5.1.*

Proof: The only difficulty takes places with verifying (iv). However, given $\phi \in \mathrm{H}_0^2(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$, we construct $\varphi = (\varphi_1, \varphi_2)$ with $\mathrm{div}\varphi = 0$ such that ϕ is the stream function associated to $\varphi = (\varphi_1, \varphi_2)$, *i.e.* $\varphi_1 = \phi_y$ and $\varphi_2 = -\phi_x$. Then we get

$$\nu \int_\Omega (\psi_{yx}\phi_{yx} + \psi_{yy}\phi_{yy} + \psi_{xx}\phi_{xx} + \psi_{xy}\phi_{xy})\, d\mathbf{x} - \int_\Omega (f_1\phi_y - f_2\phi_x)\, d\mathbf{x} = 0.$$

Integrating by parts the first and fourth terms in the first integral, where we have used the density of $\mathrm{C}_0^\infty(\Omega)$ in $\mathrm{H}_0^2(\Omega)$, we obtain the desired result.$\square$

Remark 2.5.1 *The existence and uniqueness of a weak solution* ψ *to problem (2.3.16)-(2.3.19) can be proved by different techniques without invoking to the problem (2.1.1)-(2.2.5). For some results of this nature see* e.g. *Bernis [22], Brézis and Browder [28], Grisvard [58] and J-L Lions [79].*

To establish the localization effect, as stated in Theorem 2.5.1, we will proceed as already arisen in Section 2.3. We use here the technique of integrating over a family of variable half-planes, which requires zero boundary conditions. We observe that the only non-zero boundary condition in problem (2.1.1)-(2.2.5), or problem (2.3.16)-(2.3.19), is on the boundary $x = 0$. Thus, following Bernis [18], we are lead to introduce a weighted function which will cancel the terms on this boundary. For $m \geq 2$, let

$$\psi(\mathbf{x})(x-a)_+^m = \begin{cases} 0 & \text{if } x \leq a, \\ \psi(\mathbf{x})(x-a)^m & \text{if } x > a, \end{cases}$$

where $a \geq 0$ is a variable parameter and ψ is a weak solution of (2.3.16)-(2.3.19). This function is not, in general, an admissible test function, because Ω is unbounded. Following Bernis [21], we approximate $\rho(x) = (x-a)_+^m$ by a sequence $\rho_k(x) = k^m\xi((x-a)/k)$, with $\xi \in \mathrm{C}(\mathbb{R}) \cap \mathrm{C}^2(0,\infty)$ such that $\xi \geq 0$ and

$$\xi(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^m & \text{if } 0 < x \leq \frac{1}{2}, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Due to the fact that $m \geq 2$, it is possible to show that $\psi\rho_k \in \mathrm{W}^{2,\infty}(\Omega)$ and thus $\psi(\mathbf{x})\rho_k(x)$ becomes a test function. Moreover

$$0 \leq \mathrm{D}^i\rho_1(x) \leq \mathrm{D}^i\rho_2(x) \leq \cdots \leq \mathrm{D}^i\rho_k(x) \leq \mathrm{D}^i\rho_{k+1}(x) \leq \ldots \tag{2.5.59}$$

and

$$\mathrm{D}^i\rho_k(x) \to m(m-1)\ldots[m-(i-1)](x-a)_+^{m-i}, \quad \text{as} \quad k \to \infty, \tag{2.5.60}$$

for all $x > 0$, $m \geq 2$, $0 \leq i \leq m$ and $i \leq 2$. Thus, we can prove the following result.

Lemma 2.5.2 *Let ψ be a weak solution of (2.3.16)-(2.3.19) with (2.5.58) finite. Assume that* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). Then, for every $a \geq x_g$ and every positive integer $m \geq 2$,*

$$\min(\nu,\delta)\int_\Omega \left(|\mathrm{D}^2\psi|^2 + |\psi_y|^{1+\sigma}\right)(x-a)_+^m d\mathbf{x} \leq$$

$$2m\nu\int_\Omega |\triangle\psi|\,|\psi_x|(x-a)_+^{m-1}d\mathbf{x} + 2m\nu\int_\Omega |\psi_y|\,|\psi_{xy}|(x-a)_+^{m-1}d\mathbf{x} \tag{2.5.61}$$

$$+m(m-1)\nu\int_\Omega |\triangle\psi|\,|\psi|(x-a)_+^{m-2}d\mathbf{x}.$$

Proof: Taking $\psi(\mathbf{x})\rho_k(x)$ as a test function in Definition 2.5.1, applying the Leibnitz formula and using assumptions (2.2.9) and (2.2.10), we obtain

$$\nu\int_\Omega(\triangle\psi)^2\rho_k d\mathbf{x} + \delta\int_\Omega |\psi_y|^{1+\sigma}\rho_k d\mathbf{x} \leq -2\nu\int_\Omega \triangle\psi\psi_x\rho_k' d\mathbf{x} - \nu\int_\Omega \triangle\psi\psi\rho_k'' d\mathbf{x}. \tag{2.5.62}$$

The study of the first term on the left-hand side requires an integration by parts leading to

$$\int_\Omega \psi_{xx}\psi_{yy}\rho_k\,d\mathbf{x} = \int_\Omega \psi_{xy}^2\rho_k\,d\mathbf{x} + \int_\Omega \psi_y\psi_{xy}\rho_k'\,d\mathbf{x},$$

where we have used a regularization procedure on ψ. Then from (2.5.62), it comes

$$\nu\int_\Omega |\mathrm{D}^2\psi|^2\rho_k\,d\mathbf{x} + \delta\int_\Omega |\psi_y|^{1+\sigma}\rho_k\,d\mathbf{x} \leq$$

$$-2\nu\int_\Omega \triangle\psi\psi_x\rho_k'\,d\mathbf{x} - 2\nu\int_\Omega \psi_y\psi_{xy}\rho_k'\,d\mathbf{x} - \nu\int_\Omega \triangle\psi\psi\rho_k''\,d\mathbf{x}.$$

We take the minimum on the left-hand side and apply modules. Finally, (2.5.61) follows from (2.5.59), (2.5.60) and the Lebesgue convergence theorem.□

In the first term of the left-hand side of (2.5.61), it arises the energy type term which depends on a

$$E_m(a) = \int_\Omega \left(|\mathrm{D}^2\psi|^2 + |\psi_y|^{1+\sigma}\right)(x-a)_+^m d\mathbf{x}. \tag{2.5.63}$$

We observe that $E_0(0) = E$ and a simple differentiation leads to the relations

$$\frac{dE_m(a)}{da} = -mE_{m-1}(a) \quad \text{and} \quad \frac{d^2E_m(a)}{da^2} = m(m-1)E_{m-2}(a).$$

The mentioned energy method technique, as introduced in Bernis [18], has, as main goal, to get a differential inequality for $E_m(a)$ leading to the vanishing of $E_m(a)$ and then of ψ for a large enough. The crucial part of the technique consists in to use the nonlinear structure of the equation in order to get some differential inequality. To this end, a fundamental role will be played by two inequalities. The first, is a *weighted Gagliardo-Nirenberg inequality* derived in Bernis [18] from Gagliardo-Nirenberg's inequality (1.1.5).

Lemma 2.5.3 (Bernis) *If j, k, l are integers with $0 \le j < k$, $k \ge 1$ and $l \ge 0$ and $1 \le p < \infty$ and if $1 \le r \le p$, then*

$$\int_\Omega |\mathrm{D}^j u|^p (x-a)_+^l d\mathbf{x} \le \tag{2.5.64}$$

$$C_1 \left(\int_\Omega |\mathrm{D}^k u|^p (x-a)_+^l \, d\mathbf{x} \right)^\theta \left(\int_\Omega |u|^r (x-a)_+^l \, d\mathbf{x} \right)^{p\frac{1-\theta}{r}},$$

once that the integrals of the right-hand side exist, where θ is given by

$$\frac{1}{p} = \frac{j}{2+l} + \theta \left(\frac{1}{p} - \frac{k}{2+l} \right) + (1-\theta)\frac{1}{r}$$

and $C_1 = C_1(j,k,l,p,r)$[10].

The second inequality, is a Hardy type inequality also derived in Bernis [18] from Hardy's inequality (1.1.2).

Lemma 2.5.4 (Bernis) *If $1 < p < \infty$, $l \ge 0$, $a > 0$ and u is bounded in a neighborhood of $x = a$, then*

$$\int_\Omega |u|^p (x-a)_+^l d\mathbf{x} \le \left(\frac{p}{l+1} \right)^p \int_\Omega |u_x|^p (x-a)_+^{l+p} d\mathbf{x}, \tag{2.5.65}$$

once that the integrals of both sides exist.

After the differential inequality has been obtained, the following auxiliary result, a direct consequence of Bernis [21], lead us to the conclusion.

Lemma 2.5.5 *Assume that the fractional differential inequality*

$$E_m(a) \le C_2 \left(E_{m-p}(a) \right)^\mu \tag{2.5.66}$$

holds for all $a \ge x_g > 0$, where $0 < p \le m < w$, C_2 is a positive constant and $\mu > 1$. Assume $E_{m-p}(a)$ is finite for any $a \ge x_g$. Then, the support of $E_0(a)$ is a bounded interval $[0,\ a^*]$, with $a^* \le a'$ and where

$$a' = (w-m+1) C_2^{\frac{1}{(\mu-1)(w-m)}} E^{\frac{1}{w-m}} \quad \textit{and} \quad w = \frac{\mu p}{\mu - 1}. \tag{2.5.67}$$

So, we arrive at the more difficult part of the proof of Theorem 2.5.1.

[10] For the sake of clarifying the matters, we will numerate all the constants C that appear in this proof.

Lemma 2.5.6 *Let ψ be a weak solution of (2.3.16)-(2.3.19) and assume* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). Then, the following differential inequality holds for any $a \geq x_g$, where x_g is given by (2.2.10):*

$$E_m(a) \leq C_3 \left(E_{m-2}(a)\right)^{\mu}, \quad \textit{for every integer } m > 3, \tag{2.5.68}$$

where $C_3 = C_3(L, m, \delta, \nu, \sigma)$ and $\mu = \mu(m, \sigma)$ are positive constants, with $\mu > 1$. Moreover, $E_2(a) < \infty$ for any $a \geq x_g$. In fact,

$$E_2(a) \leq C_4 E_0(a) + C_5 \left(E_0(a)\right)^{\mu}; \tag{2.5.69}$$

where C_4 and C_5 are positive constants, with C_4 an universal constant and $C_5 = C_5(L, \delta, \nu, \sigma)$.

Proof: We rewrite (2.5.61) as

$$\min(\nu, \delta) E_m(a) \leq 2m\nu I_1 + 2m\nu I_2 + m(m-1)\nu I_3. \tag{2.5.70}$$

Now, let us estimate I_1, I_2 and I_3 in terms of $E_m(a)$ and $E_{m-2}(a)$.

Assume $m > 3$. Applying Cauchy's inequality with ε to each one of these terms and then adding up the connected terms, (2.5.70) comes

$$\min(\nu, \delta) E_m(a) \leq m(m+1)\nu\varepsilon E_m(a) + \frac{m\nu}{\varepsilon} I_{22} + \frac{m(m-1)\nu}{2\varepsilon} I_{32}, \tag{2.5.71}$$

where

$$I_{22} := \int_{\Omega} |\nabla\psi|^2 (x-a)_+^{m-2} d\mathbf{x} \quad \text{and} \quad I_{32} := \int_{\Omega} \psi^2 (x-a)_+^{m-4} d\mathbf{x}. \tag{2.5.72}$$

Applying the weighted Gagliardo-Nirenberg inequality (2.5.64) to I_{22} for the function ψ , with $j = 1$, $k = 2$, $l = m-2$, $p = 2$ and $r = 1 + \sigma$, we get

$$I_{22} \leq C_6 \left(\int_{\Omega} |\mathrm{D}^2\psi|^2 (x-a)_+^{m-2} \, d\mathbf{x} \right)^{\theta} \left(\int_{\Omega} |\psi|^{1+\sigma} (x-a)_+^{m-2} \, d\mathbf{x} \right)^{2\frac{1-\theta}{1+\sigma}},$$

where

$$\theta = \frac{2(1+\sigma) + (1-\sigma)m}{4(1+\sigma) + (1-\sigma)m} \tag{2.5.73}$$

and $C_6 = C_6(m, \sigma)$ ($C_6 = C_1$ with $j = 1$, $k = 2$, $l = m-2$, $p = 2$ and $r = 1+\sigma$). Notice that $0 < \theta < 1$, because $0 < \sigma < 1$. Applying Poincaré's inequality, with $p = 1 + \sigma$, to the term $\int_0^L |\psi|^{1+\sigma} dy$ and then the algebraic inequality (1.1.1),

$$I_{22} \leq C_7 \left(E_{m-2}(a)\right)^{\mu}, \quad C_7 = C_7(L, m, \sigma), \tag{2.5.74}$$

where

$$0 < \sigma < 1 \Rightarrow \mu = 1 + \frac{2(1-\sigma)}{4(1+\sigma) + (1-\sigma)m} > 1. \tag{2.5.75}$$

Now, applying the Hardy type inequality (2.5.65) to I_{32} for the function ψ, with $l = m-4$ and $p = 2$

$$I_{32} \leq \left(\frac{2}{m-3}\right)^2 \int_{\Omega} |\psi_x|^2 (x-a)_+^{m-2} d\mathbf{x}.$$

Because $|\psi_x|^2 \leq |\nabla\psi|^2$, $I_{32} \leq 4/(m-3)^2 I_{22}$, then from (2.5.74)

$$I_{32} \leq \frac{4C_7}{(m-3)^2} \left(E_{m-2}(a)\right)^{\mu}, \tag{2.5.76}$$

where θ and μ are given by (2.5.73) and (2.5.75), respectively. Then, from (2.5.74) and (2.5.76), (2.5.71) comes

$$\min(\nu,\delta) E_m(a) \leq \varepsilon C_8 E_m(a) + \frac{1}{\varepsilon} C_9 \left(E_{m-2}(a)\right)^{\mu},$$

where $C_8 = C_8(m,\nu)$, $C_9 = C_9(L,m,\nu,\sigma)$ and $m > 3$. Choosing $\varepsilon = \min(\nu,\delta)/(2C_8)$, we achieve to the fractional inequality (2.5.68), which is valid for every $m > 3$ and where

$$C_3 = 4C_7 \frac{m^2(m+1)(m^2-4m+7)}{(m-3)^2} \frac{\nu^2}{\min^2(\nu,\delta)} \tag{2.5.77}$$

and C_7, given immediately after to (2.5.73), is the constant resulting from Poincaré's and Gagliardo-Nirenberg's (2.5.64) inequalities.

In order to prove that $E_2(a) < \infty$ for any $a \geq x_g$, we have only to worry about with the term I_3, because for the other terms the above estimates remain valid. For $m = 2$,

$$I_3 = \int_{\Omega_a} \triangle\psi\, \psi\, d\mathbf{x}.$$

Applying Cauchy's inequality with a suitable ε to I_3 and proceeding in the same manner for the other terms as in the preceding case, but taking $m = 2$, we obtain the analogous inequality to (2.5.71)

$$\min(\nu,\delta) E_2(a) \leq 4\nu\varepsilon E_2(a) + 2\nu\varepsilon E_0(a) + \frac{2\nu}{\varepsilon} I_{22(m=2)} + \frac{\nu}{\varepsilon} I_{32(m=2)}, \tag{2.5.78}$$

where

$$I_{22(m=2)} := \int_{\Omega_a} |\nabla\psi|^2 \, d\mathbf{x} \quad \text{and} \quad I_{32(m=2)} := \int_{\Omega_a} \psi^2 \, d\mathbf{x}. \tag{2.5.79}$$

Taking $m = 2$ in (2.5.74),

$$I_{22(m=2)} \leq C_{10} \left(E_0(a)\right)^{\mu}, \quad C_{10} = C_{10}(L,\sigma), \tag{2.5.80}$$

where, from (2.5.73) and (2.5.75), respectively,

$$\theta = \frac{2}{3+\sigma} \quad \text{and} \quad \mu = \frac{4}{3+\sigma}, \tag{2.5.81}$$

and $C_{10} = C_7$ with $m = 2$. Applying Poincaré's inequality to the term $\int_0^L |\psi|^2 \, dy$ of $I_{32(m=2)}$, $I_{32(m=2)} \leq C_{11} I_{22(m=2)}$, $C_{11} = C_{11}(L)$, and from (2.5.80),

$$I_{32(m=2)} \leq C_{12} \left(E_0(a)\right)^{\mu}, \quad C_{12} = C_{10} C_{11}. \tag{2.5.82}$$

Then, (2.5.80) and (2.5.82) yield that (2.5.78) comes

$$\min(\nu, \delta) E_2(a) \leq \varepsilon C_{13} (E_2(a) + E_0(a)) + \frac{1}{\varepsilon} C_{14} \left(E_0(a)\right)^{\mu}, \tag{2.5.83}$$

where $C_{13} = C_{13}(\nu)$, $C_{14} = C_{14}(L, \nu, \sigma)$ are positive constants. Now, if we choose $\varepsilon = \min(\nu, \delta)/(2C_{13})$ in (2.5.83), we come to the differential inequality (2.5.69), where

$$C_4 = \frac{1}{2}, \quad C_5 = 16 C_{10} \frac{\nu^2}{\min^2(\nu, \delta)} (2 + C_{11}),$$

with θ and μ given by (2.5.81) and C_{10} and C_{11} immediately after. Now, since $E_0(a) < \infty$ for any $a \geq x_g$, we get that $E_2(a)$ is also finite for any $a \geq x_g$.□

Proof: [Theorem 2.5.1] Taking $m = 4$ in Lemma 2.5.6, we have the fractional differential inequality

$$E_4(a) \leq C_{15} \left(E_2(a)\right)^{\mu}, \tag{2.5.84}$$

where from (2.5.73), (2.5.75) and (2.5.77), respectively,

$$\theta = \frac{3 - \sigma}{4}, \quad \mu = \frac{5 - \sigma}{4} \quad \text{and} \quad C_{15} = C_5 \text{ with } m = 4. \tag{2.5.85}$$

One can easily see that, due to $0 < \sigma < 1$, we have $1/2 < \theta < 3/4$ and $1 < \mu < 5/4$. By Lemma 2.5.6, $E_2(a)$ is finite. Then from Lemma 2.5.5, with $m = 4$, $p = 2$ and $w = 8/(1 - \sigma) + 2 > 4 = m$, the support of $E_0(a)$ is a bounded interval $[0, a^*]$ with $a^* \leq a'$ and where, from (2.5.67) and (2.5.85),

$$a' = \frac{7 + \sigma}{1 - \sigma} C_{15}^{\frac{2}{3+\sigma}} E^{\frac{1-\sigma}{2(3+\sigma)}}.$$

Then

$$E_0(a) = \int_{\Omega_a} \left(|\nabla \mathbf{u}|^2 + |u|^{1+\sigma}\right) d\mathbf{x} = 0$$

for $a > a'$, which implies $u = 0$ and v is constant almost everywhere for $x > a'$. Finally, from (2.2.4), $v = 0$ in the same domain.□

Remark 2.5.2 *Obviously, we obtain an analogous localization effect if we replace the role of variables x and y for the study of unbounded sets of the form $\Omega = (0, L) \times (0, \infty)$, and if we modify correspondingly the conditions (2.2.9) and (2.2.10).*

Remark 2.5.3 *We obtain the same localization effect if we consider the non-constant semi-infinite strip $\Omega = (0, \infty) \times (L_1(x), L_2(x))$, with $L_1,\ L_2 \in \mathrm{C}^2(0, \infty)$, $k_1 \leq |L_2(x) - L_1(x)| \leq k_2$, $|L_1'(x)|$, $|L_2'(x)| \leq k_3$, and $|L_1''(x)|$, $|L_2''(x)| \leq k_4$ for all $x \geq 0$, where k_i, $i = 1, \ldots, 4$, are positive constants (see the proof in Antontsev et al [8]).*

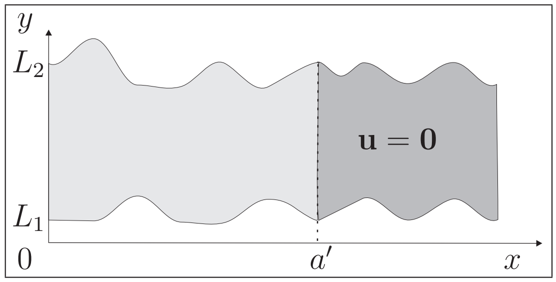

Figure 2.3: Non-constant semi-infinite strip.

Remark 2.5.4 *The localization effect can be extended to the limit case of $\sigma = 0$. But, in the case of $\sigma = 1$, the above arguments lead to the inequality $E_4(a) \leq C\, E_2(a)$ (see (2.5.84)) and then we can only derive the exponential decay*

$$E_2(a) \leq Ce^{-Ca} \quad \textit{for all } a > 0, \tag{2.5.86}$$

where C means two different positive constants (see Bernis [19]). An exponential decay of this kind was obtained separately by Knowles [68] and Toupin [117] in their energy approach to the investigation of the Saint-Venant's Principle[11] *in classical elasticity theory by using differential inequality techniques. There it was considered equation (2.3.16) with $\mathbf{f} = \mathbf{0}$ in a simply connected bounded planar domain Ω (a cross section of a cylindrical body).*

2.6 Stagnation Effect

In the arguments we have considered in the previous section, one can realize that the parameter a we have chosen is such that $a \geq x_g$, with x_g given in (2.2.10). To work with $a < x_g$, we have to assume an extra condition on the second component of the forces field. Let us assume the body forces field satisfy (2.2.9), with $g \equiv 0$, and

$$|f_2(\mathbf{x}, \mathbf{u})| \leq \gamma(x_s - x)_+^{\zeta} \tag{2.6.87}$$

for some $x_s > 0$, where γ and ζ are positive constants, with ζ to be specified later on. Because of this last condition, we say the second component of the body forces field has a *stagnation line* at $x = x_s$. The existence and uniqueness of a weak solution for this case, is guaranteed by Theorem 2.4.1, where $\mathbf{h}(\mathbf{x}, \mathbf{u}) = \mathbf{0}$, and Theorem 2.4.3, respectively.

Theorem 2.6.1 *There exist some positive constants γ and ζ such that if (2.2.9), with $g \equiv 0$, and (2.6.87) hold, then $\mathbf{u} = \mathbf{0}$ for $x \geq x_s$ and any $y \in (0, L)$.*

[11] : Two systems of stresses in a plane region Ω, corresponding to the two statically equivalent surface tractions on the boundary $\partial\Omega$ would be approximately the same at points of Ω which are remote from the arc $\mathcal{C} \subset \partial\Omega$ on which the two distributions of surface traction differ in detail (cf. Knowles [68]).

Proof: If we consider $a \geq x_s$, we fall in the conditions studied in Theorem 2.5.1 with $g = 0$. Let us assume $a < x_s$. Then we have to add to the right-hand side of (2.5.61) the terms $\gamma(mK_1 + K_2)$ resulting from condition (2.6.87) (see also (2.3.20)), where

$$K_1 = \int_\Omega (x_s - x)_+^\zeta |\psi|(x-a)_+^{m-1} d\mathbf{x}, \; K_2 = \int_\Omega (x_s - x)_+^\zeta |\psi_x|(x-a)_+^m d\mathbf{x}. \tag{2.6.88}$$

This lead us to the counterpart of (2.5.68)[12]

$$E_m(a) \leq C_1 \left(E_{m-2}(a)\right)^\mu + C_2 E_{m,2\zeta}(a), \tag{2.6.89}$$

for every integer $m > 3$, where $C_i = C_i(L, m, \delta, \gamma, \nu, \sigma)$, $i = 1, 2$, μ is given by (2.5.75) and

$$E_{m,\zeta}(a) = \int_\Omega (x_s - x)_+^\zeta (x-a)_+^m d\mathbf{x}, \quad m \geq 2. \tag{2.6.90}$$

Taking $m = 4$ in (2.6.89) and using an integration by parts on (2.6.90), we arrive at the counterpart of (2.5.84), where we put $E_4(a) = z(a)$,

$$z(a) \leq C_3 \left(z''(a)\right)^\mu + C_4 (x_s - a)_+^{2\zeta+4}, \tag{2.6.91}$$

where $C_3 = C_3(L, \delta, \gamma, \nu, \sigma)$ and $C_4 = C_4(\zeta, L, \delta, \gamma, \nu, \sigma)$. The solutions of (2.6.91), with $\zeta = (2-\mu)/(\mu-1)$ (we notice that from (2.5.85), $\zeta > 3$), are of the form $z(a) = C_5 (x_s - a)_+^{\frac{2\mu}{\mu-1}}$ with the positive constant C_5 satisfying

$$C_5 \leq C_3 C_5^\mu \left[\frac{2\mu(\mu+1)}{(\mu-1)^2}\right]^\mu + C_4.$$

Then $E_4(a) = C_5 (x_s - a)_+^{\frac{2\mu}{\mu-1}}$ and consequently $\mathbf{u} = \mathbf{0}$ for $x \geq x_s$.□

Remark 2.6.1 *From the Physics point of view, the localization property proved in the above theorem means the fluid stops at the same stagnation line as the second component of the body forces. That is the reason why we call this localization effect as the stagnation effect.*

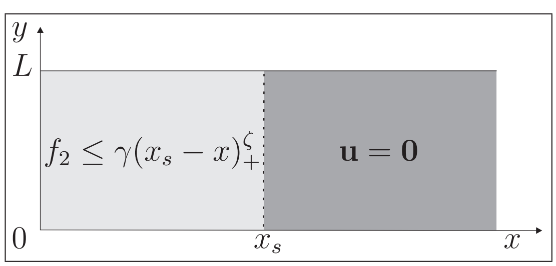

Figure 2.4: Stagnation effect.

[12] Again, for clarifying the exposition, we will numerate all the constants appearing in this proof.

2.7 Generalizations

We can generalize the results of the foregoing sections, if we consider a localized forces field, *i.e.* if besides (2.2.9) and (2.2.10), $\mathbf{f}$ satisfies

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{0} \quad \text{for} \quad x \geq x_{\mathbf{f}} \quad \text{and} \quad y \in (0, L),$$

for some $x_{\mathbf{f}}$ such that $0 \leq x_g \leq x_{\mathbf{f}}$ (x_g given in (2.2.10)). In this case (2.2.9) and (2.2.10) are replaced by

$$-\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \geq \delta \, \chi_{\mathbf{f}}(\mathbf{x}) \, |u|^{1+\sigma} - g(\mathbf{x}), \tag{2.7.92}$$

where δ and σ are given in (2.2.9), $\chi_{\mathbf{f}}$ is the characteristic function of the interval $(0, x_{\mathbf{f}})$ and g is a scalar function satisfying (2.2.10) (cf. Antontsev et al [7, 8]).

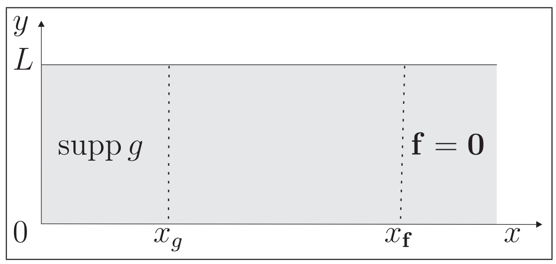

Figure 2.5: Localized forces field.

The existence results stated in Theorems 2.4.1 and 2.4.2 as well the uniqueness result stated in Theorem 2.4.3 remain valid here. We need only to consider the following explicit form of the localized forces field

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = -\delta \, \chi_{\mathbf{f}}(|u|^{\sigma-1}u, 0) - \mathbf{h}(\mathbf{x}, \mathbf{u}), \tag{2.7.93}$$

and the localized global energy

$$E := \int_{\Omega} \left(|\nabla \mathbf{u}|^2 + \chi_{\mathbf{f}} |u|^{1+\sigma} \right) \, d\mathbf{x} \tag{2.7.94}$$

instead of (2.4.21) and (2.5.58), respectively.

The stopping effect stated in Theorem 2.5.1 can be generalized by the following one.

Theorem 2.7.1 *Assume that* $\mathbf{f}$ *satisfies (2.7.92) and (2.2.10). Then:*
(i) if $x_{\mathbf{f}} = \infty$, $\mathbf{u}$ *is any weak solution of (2.1.1)-(2.2.5), with finite energy (2.7.94), then* $\mathbf{u}(x, y) = \mathbf{0}$ *for* $x > a'$, *where* $a' = a'(E, L, \delta, \nu, \sigma)$ *is a positive constant;*
(ii) if $x_{\mathbf{f}} < \infty$, *then there exists, at least, one weak solution* $\mathbf{u}$ *of (2.1.1)-(2.2.5), with a finite energy (2.7.94), such that if* $a' < x_{\mathbf{f}}$, *then* $\mathbf{u}(x, y) = \mathbf{0}$ *for* $x > a'$;
(iii) if, in addition, we assume $\mathbf{f}$ *non-increasing (condition (2.4.56)), then conclusion (ii) holds for the unique solution of (2.1.1)-(2.2.5).*

Proof: If $x_{\mathbf{f}} = \infty$, we fall in the case of Theorem 2.5.1. We need only to replace the energy type terms (2.5.63) by

$$E_m(a) = \int_\Omega \left(|\mathrm{D}^2\psi|^2 + \chi_{\mathbf{f}}|\psi_y|^{1+\sigma}\right)(x-a)_+^m d\mathbf{x}. \tag{2.7.95}$$

Assume now that $x_{\mathbf{f}} < \infty$. Then we construct a weak solution in the following way

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \widetilde{\mathbf{u}}(\mathbf{x}) & \text{if} \quad x < a' \\ \mathbf{0} & \text{if} \quad x \geq a', \end{cases}$$

with $\widetilde{\mathbf{u}}(\mathbf{x})$ weak solution of (2.1.1)-(2.2.5), where $\mathbf{f}$ satisfies (2.2.10) and (2.7.92) with $x_{\mathbf{f}} = \infty$. By the proof of the above case and the assumption $a' < x_{\mathbf{f}}$, we get that $\mathbf{u}(\mathbf{x})$ is a weak solution of the original problem, *i.e.* where $\mathbf{f}$ satisfies (2.2.10) and (2.7.92) with $x_{\mathbf{f}} < \infty$.□

The stagnation effect stated in Theorem 2.6.1 can also be generalized for a localized forces field. In this case, we assume the body forces field $\mathbf{f}$ satisfies (2.6.87) and

$$-\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \geq \delta\, \chi_{\mathbf{f}}(\mathbf{x})\, |u|^{1+\sigma}, \tag{2.7.96}$$

where δ and σ are given in (2.2.9). The proof of this case is entirely analogous of the proof of Theorem 2.6.1 (cf. Antontsev et al [9]).

Chapter 3

Stationary Navier-Stokes Problem

> *"Uniqueness occurs only when the data are small enough, or the viscosity is large enough."*
>
> R. Temam [115, p. 157].

In this chapter the results we have proven in the precedent chapter are extended to the case of the stationary Navier-Stokes problem in the same domain and with the same boundary conditions. We present the problem in Section 3.1 and we prove the existence and uniqueness results in Section 3.2. In this case, as it is known by the specialists in the field of Mathematical Fluid Mechanics, uniqueness holds only for small data or large viscosity, which corresponds to consider a small Reynolds number. As in Chapter 2, the existence and uniqueness theorems are proved by using some known results for the Navier-Stokes equations with prescribed linear forces field and, in almost all the situations, the reader is addressed to the the monograph by Galdi [50] to see the proofs of these results. But in fact and again the monographs by Constantin and Foias [33], Ladyzhenskaya [72] and Temam [115] were also used. The localizations effects are proved in Section 3.3, where many estimates obtained in Chapter 2 can be used. Although this help, the problem of obtaining the same localization effects is much more technical in this case. This is because of terms resulting from the convection which cause some difficulties if one does not want to obtain estimates depending on the norms of the function involved. This problem is overcame with the help of a result proved in an appendix section (Section 3.4).

3.1 Introduction

We consider the *stationary Navier-Stokes system*

$$-\nu\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{f}-\nabla p, \tag{3.1.1}$$

$$\operatorname{div}\mathbf{u}=\mathbf{0} \tag{3.1.2}$$

in the semi-infinite strip $\Omega=(0,\infty)\times(0,L)$, $L>0$ a positive constant, where ν is the *kinematic viscosity coefficient*. Navier-Stokes system (3.1.1)-(3.1.2) is derived from Navier-Stokes system {(1.4.22), (1.4.11), (1.4.23)} by assuming the fluid is homogeneous, the kinematic viscosity is constant and the velocity and pressure do not depend explicitly on time. As already mentioned in Section 2.1, the consideration of the planar domain Ω, means that we are facing a flow with only two velocity components: $u(x,y)$ and $v(x,y)$.

We consider the Navier-Stokes system (3.1.1)-(3.1.2) appended with the boundary conditions

$$\mathbf{u}(0,y)=\mathbf{u}_*(y),\ y\in(0,L), \tag{3.1.3}$$

$$\mathbf{u}(x,0)=\mathbf{u}(x,L)=\mathbf{0},\ x\in(0,\infty), \tag{3.1.4}$$

and the velocity at infinity

$$|\mathbf{u}(x,y)|\to 0,\ \text{as } x\to\infty \text{ and } \ y\in(0,L). \tag{3.1.5}$$

We assume also the compatibility conditions (2.2.6)-(2.2.7) and that the body forces field is given in a feedback dissipative form, $\mathbf{f}:\Omega\times\mathbb{R}^2\to\mathbb{R}^2$, $\mathbf{f}=(f_1,f_2)$, such that for every $\mathbf{u}\in\mathbb{R}^2$ and almost all $\mathbf{x}\in\Omega$ (2.2.9) and (2.2.10) hold.

We recall the well known result asserting that, if $\mathbf{f}\equiv\mathbf{0}$, or $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and has compact support in Ω, the weak solutions $\mathbf{u}$ of problem (3.1.1)-(3.1.5) have an exponential decay as (2.2.8) which is optimal (cf. Galdi [50, §XI.4]). See also Horgan [62] for plane entry flows and Amick and Fraenkel [3] for an exhaustive study on plane flows in channels of various types.

Introducing the stream function (2.3.15), we reduce the study of problem (3.1.1)-(3.1.5) to the following one

$$\nu\triangle^2\psi+\tfrac{\partial f_1}{\partial y}-\tfrac{\partial f_2}{\partial x}=\psi_y\triangle\psi_x-\psi_x\triangle\psi_y \quad \text{in } \Omega, \tag{3.1.6}$$

$$\psi(x,0)=\psi(x,L)=\tfrac{\partial\psi}{\partial n}(x,0)=\tfrac{\partial\psi}{\partial n}(x,L)=0 \quad \text{for } x\in(0,\infty), \tag{3.1.7}$$

$$\psi(0,y)=\textstyle\int_0^y u_*(s)ds,\ \tfrac{\partial\psi}{\partial n}(0,y)=v_*(y) \quad \text{for } y\in(0,L), \tag{3.1.8}$$

$$\psi(x,y),\ |\nabla\psi(x,y)|\to 0 \text{ as } x\to\infty \quad \text{and for } y\in(0,L), \tag{3.1.9}$$

where the pressure term does not appear anymore (see *e.g.* Ladyzhenskaya [72, §5.5], and Quarteroni and Valli [96, §10.4]) and, as in Section 2.5, $\mathbf{f}=(f_1,f_2)=(f_1(\mathbf{x},\psi_y,-\psi_x),\ f_2(\mathbf{x},\psi_y,-\psi_x))$. See the considerations given in the paragraph before Definition 2.5.1 on page 34. Proceeding as in the final part of Section 2.3, we obtain

$$\int_\Omega(\triangle\psi)^2\,(x-a)_+^m d\mathbf{x}-\int_\Omega f_1\psi_y(x-a)_+^m d\mathbf{x}=-m\int_\Omega\triangle\psi\,\psi_y\,\psi\,(x-a)_+^{m-1}d\mathbf{x}$$

$$+\int_\Omega f_2\left(m\psi(x-a)_+^{m-1}+\psi_x(x-a)_+^m\right)\,d\mathbf{x} \tag{3.1.10}$$

$$-\nu m\int_\Omega \triangle\psi\left(2\psi_x(x-a)_+^{m-1}+(m-1)\psi(x-a)_+^{m-2}\right)\,d\mathbf{x}.$$

3.2 Weak Formulation

In this section we consider the forces field given by (2.4.21)-(2.4.22) and the solenoidal vector spaces $\widetilde{\mathbf{H}}(\Omega)$ and $\widetilde{\mathbf{H}}_0(\Omega)$ defined in Section 2.4. As in that section, we shall search for solutions $\mathbf{u}$ satisfying (3.1.5) and (2.4.23).

Definition 3.2.1 *A vector function* $\mathbf{u}$ *is a weak solution of problem (3.1.1)-(3.1.5), if:*
(i) $\mathbf{u}\in\widetilde{\mathbf{H}}(\Omega)$, $\mathbf{f}(\mathbf{x},\mathbf{u})\in\mathbf{L}^1_{\text{loc}}(\Omega)$*;*
(ii) For every $\varphi\in\widetilde{\mathbf{H}}_0(\Omega)\cap\mathbf{L}^\infty(\Omega)$ *with compact support,*

$$\nu\int_\Omega \nabla\mathbf{u}:\nabla\varphi\,d\mathbf{x}+\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\varphi\,d\mathbf{x}=\int_\Omega \mathbf{f}\cdot\varphi\,d\mathbf{x}.$$

As in Section 2.4, we can establish an existence result under the growth condition (2.4.24). Here we need an extra condition to guarantee the uniqueness of solutions for the auxiliary problems we have to consider in the curse of the proof. Because of the nonlinearity of the Navier-Stokes equations (3.1.1), we know that uniqueness holds only if the coefficient of kinematical viscosity ν is sufficiently large or if the data are sufficiently small. This corresponds to say that uniqueness holds only if the dimensionless Reynolds number is sufficiently small (see *e.g.* Galdi [50, §VIII.2]).

Theorem 3.2.1 *Let us assume* $\mathbf{u}_*\in\mathbf{H}^{\frac{1}{2}}(0,L)$, $\mathbf{f}(\mathbf{x},\mathbf{u})$ *satisfies (2.4.21), (2.4.22) and the growth condition (2.4.24) holds. In addition, we assume that the problem (3.1.1)-(3.1.5) with* $\mathbf{f}(\mathbf{x},\mathbf{u})=\mathbf{f}(\mathbf{x},0)$ *has a unique weak solution in* $\Omega^R=(0,R)\times(0,L)$, *for every* $R>0$[1]*. Then, there exists, at least, one weak solution* $\mathbf{u}$ *of problem (3.1.1)-(3.1.5). Moreover,* $\mathbf{f}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}$ *lies in* $\mathrm{L}^1(\Omega)$ *and* $\mathbf{u}$ *satisfies to the energy estimate (2.4.25).*

Proof: The proof follows exactly the same steps of the proof of Theorem 2.4.1.
First step. For a given $N\in\mathbb{N}$, we start by considering the auxiliary problem in $\Omega^N=(0,N)\times(,L)$

$$-\nu\triangle\mathbf{u}^N+(\mathbf{u}^N\cdot\nabla)\mathbf{u}^N=\mathbf{f}(\mathbf{x},\mathbf{u}^N)-\nabla p^N\quad\text{in }\Omega^N, \tag{3.2.11}$$

$$\operatorname{div}\mathbf{u}^N=0\quad\text{in }\Omega^N, \tag{3.2.12}$$

$$\mathbf{u}^N=\mathbf{u}_*(y)\text{ for }x=0 \tag{3.2.13}$$

$$\mathbf{u}^N=\mathbf{0}\text{ for }x=N\text{ and }y=0,\ L. \tag{3.2.14}$$

[1]It is implied we are considering problem (3.1.1)-(3.1.5) in $\Omega^R=(0,R)\times(0,L)$ with (3.1.5) replaced by $\mathbf{u}(R,y)=\mathbf{0}$ for $y\in(0,L)$.

With no lost of generality, we assume $N > 1$ and let $\mathbf{U}^1$ be an extension of $\mathbf{u}_*$ to $\Omega^1 = (0,1) \times (0,L)$ verifying (i)-(iii) of (2.4.30). One can prove (see *e.g.* Galdi [50, §VIII.4]) that for any $\alpha > 0$, there exists an extension $\mathbf{U}^1$ satisfying (i)-(iii) of (2.4.30) and

$$\left| \int_{\Omega^1} \mathbf{v} \cdot \nabla \mathbf{U}^1 \cdot \mathbf{v}\, d\mathbf{x} \right| \leq \alpha \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega^1)} \quad \text{for all } \mathbf{v} \in \mathbf{H}^1(\Omega^1).$$

Now, we consider the extension $\mathbf{U}^N$ to Ω^N such that $\mathbf{U}^N = \mathbf{U}^1$ if $x < 1$ and $\mathbf{U}^N = \mathbf{0}$ if $x \geq 1$. From what we have said above, $\mathbf{U}^N \in \mathbf{H}^1(\Omega^N)$, $\mathbf{U}^N$ satisfies (2.4.32) and

$$\left| \int_{\Omega^N} \mathbf{v} \cdot \nabla \mathbf{U}^N \cdot \mathbf{v}\, d\mathbf{x} \right| \leq \alpha \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega^N)} \quad \text{for all } \ \mathbf{v} \in \mathbf{H}^1(\Omega^N). \tag{3.2.15}$$

Second step. We look for solutions $\mathbf{u}^N$ of the form $\mathbf{u}^N = \mathbf{w}^N + \mathbf{U}^N$, where $\mathbf{U}^N$ is the extension given in the *First step* and $\mathbf{w}^N$ solves the problem in Ω^N

$$-\nu \triangle \mathbf{w}^N + (\mathbf{w}^N \cdot \nabla)\mathbf{w}^N = \tag{3.2.16}$$

$$\mathbf{f}(\mathbf{x}, \mathbf{w}^N + \mathbf{U}^N) + \nu \triangle \mathbf{U}^N - (\mathbf{U}^N \cdot \nabla)\mathbf{U}^N - (\mathbf{w}^N \cdot \nabla)\mathbf{U}^N - (\mathbf{U}^N \cdot \nabla)\mathbf{w}^N - \nabla p^N,$$

$$\operatorname{div} \mathbf{w}^N = 0, \tag{3.2.17}$$

$$\mathbf{w}^N = \mathbf{0} \quad \text{for } x = 0,\ N \text{ and } y = 0,\ L. \tag{3.2.18}$$

Proceeding exactly as in the correspondent step of the proof of Theorem 2.4.1 and using some known existence results for the Navier-Stokes problem with linear forces field (see *e.g.* Galdi [50, §IX.4]), we prove the existence of, at least, one weak solution of problem (3.2.16)-(3.2.18) and consequently the existence of, at least, one weak solution of problem (3.2.11)-(3.2.14), with $\mathbf{f}(\mathbf{x}, \mathbf{u}^N)$.

Third step. In this step is proved the estimate (2.4.41) is independent of N just in the same way we did in *Third step* of the proof of Theorem 2.4.1.

Fourth step. In this step we can make a transcription, *mutatis mutandis*, of the correspondent step of the proof of Theorem 2.4.1.□

Here we can also establish an existence result by dropping the growth condition (2.4.24), but then we have to assume the vectors angle condition (2.4.43) and, furthermore, the condition (2.4.42).

Theorem 3.2.2 *Theorem 3.2.1 remains valid if we replace the growth condition (2.4.24) by the vectors angle condition (2.4.43) and we assume also condition (2.4.42). Here the energy estimate takes the form of (2.4.44).*

Proof: *First step.* All that is written in the *First step* of the proof of Theorem 3.2.1 is valid here.

Second step. We consider, first, the intermediary case in which we assume, additionally, (2.4.46).

If we consider the problem (3.2.11)-(3.2.14), with $\mathbf{f}(\mathbf{x},\mathbf{u}^N)$ replaced by $\mathbf{f}(\mathbf{x})$ given arbitrarily, for instance $\mathbf{f}\in\mathbf{L}^2(\Omega^N)$, then we know the existence of, at least, one weak solution $\mathbf{u}^N\in\mathbf{H}^1(\Omega^N)$ (see *e.g.* Galdi [50, §IX.4]), which satisfies to the energy relation

$$\nu\int_{\Omega^N}\nabla\mathbf{u}^N:\nabla(\mathbf{u}^N-\mathbf{U}^N)\,d\mathbf{x}+\int_{\Omega^N}\mathbf{u}^N\cdot\nabla\mathbf{u}^N\cdot(\mathbf{u}^N-\mathbf{U}^N)d\mathbf{x}= \tag{3.2.19}$$

$$\int_{\Omega^N}\mathbf{f}\cdot(\mathbf{u}^N-\mathbf{U}^N)\,d\mathbf{x}.$$

Proceeding, *e.g.* as in the last reference, using (2.4.32) and (3.2.15), one can prove the following estimate

$$\left\|\mathbf{u}^N\right\|^2_{\mathbf{H}^1(\Omega^N)}\le C,\quad C=C\left(L,\nu,\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)},\|\mathbf{f}\|_{\mathbf{L}^2(\Omega^N)}\right).$$

Using the Leray-Schauder fixed point theorem in the same manner such as we did in the proof of Theorem 2.4.2, we prove the existence of, at least, one weak solution $\mathbf{u}^N\in\mathbf{H}^1(\Omega^N)$ to the problem (3.2.11)-(3.2.14), with $\mathbf{f}(\mathbf{x},\mathbf{u}^N)$.

Third step. In the energy relation (3.2.19) satisfied by $\mathbf{u}^N$, we use the assumption (2.4.21), next we add $\left|\mathbf{h}(\mathbf{x},\mathbf{u}^N)\cdot\mathbf{u}^N\right|$ to both sides of the resultant equation, we use assumptions (2.2.10), (2.4.46), (2.4.51) and we use Hölder's inequality. Then, we use (3.2.15), the Sobolev imbedding (2.4.39) and we apply Young's inequality with a suitable $\varepsilon>0$, to obtain the *a priori* estimate (2.4.50), independent of N, for $\mathbf{u}^N$.

Fourth Step. Keeping in mind the assumption (2.4.46) and proceeding exactly as in the proof of the correspondent step of the proof of Theorem 2.4.2, we prove the existence of, at least, one weak solution $\mathbf{u}\in\widetilde{\mathbf{H}}(\Omega)$ to the problem (3.1.1)-(3.1.5) and which satisfies the energy estimate (2.4.44).

Fifth step. To proceed with the general case, *i.e.* dropping condition (2.4.46), we use the same truncation and approximation argument such as we did in the *Fifth step* of the proof of Theorem 2.4.2.□

To prove the uniqueness of weak solution, we assume the non-increasing condition (2.4.56) and because of the nonlinearity of the Navier-Stokes equations, we assume that ν and $\mathbf{u}_*$ are such that the problem (3.1.1)-(3.1.5), with $\mathbf{f}(\mathbf{x},\mathbf{0})$, has a unique weak solution in $\widetilde{\mathbf{H}}(\Omega)$.

Theorem 3.2.3 *Let* $\mathbf{u}_1$*,* $\mathbf{u}_2$ *be two weak solutions of (3.1.1)-(3.1.5). Assume (2.4.56) holds and that the problem (3.1.1)-(3.1.5), with* $\mathbf{f}(\mathbf{x},\mathbf{0})$*, has a unique weak solution in* $\widetilde{\mathbf{H}}(\Omega)$*. Then* $\mathbf{u}_1=\mathbf{u}_2$*.*

Proof: Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two weak solutions. Then, according to Definition 3.2.1, $\mathbf{u_1}-\mathbf{u_2}\in\mathbf{H}^1_0(\Omega)$ and

$$\nu\int_{\Omega}\left|\nabla(\mathbf{u}_1-\mathbf{u}_2)\right|^2\,d\mathbf{x}+\int_{\Omega}(\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla\mathbf{u}_1\cdot(\mathbf{u}_1-\mathbf{u}_2)\,d\mathbf{x} \tag{3.2.20}$$

$$= \int_\Omega (\mathbf{f}(\mathbf{x},\mathbf{u}_1) - \mathbf{f}(\mathbf{x},\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \; d\mathbf{x}.$$

Now, proceeding as in the proof of Theorem 2.4.3, we prove

$$\nu \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \; d\mathbf{x} + \int_\Omega (\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{u}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \; d\mathbf{x} \leq 0.$$

Then arguing as for the proof of the uniqueness result with $\mathbf{f}(\mathbf{x},\mathbf{0})$ (see *e.g.* Galdi [50]), we get that

$$\int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, d\mathbf{x} = 0$$

and from Poincaré's inequality, we obtain the result.□

Remark 3.2.1 *The assumption of a unique weak solution for problem (3.1.1)-(3.1.5), with* $\mathbf{f}(\mathbf{x},\mathbf{0})$*, is fundamental to prove the uniqueness of weak solutions for general Navier-Stokes problems with prescribed forces field. This assumption is equivalent to assume*

$$-\int_\Omega \varphi \cdot \nabla \mathbf{u} \cdot \varphi \, d\mathbf{x} < C \|\nabla \varphi\|^2_{\mathbf{L}^2(\Omega)}, \quad \text{with} \quad C < \nu,$$

for every $\mathbf{u} \in \widetilde{\mathbf{H}}$ *and* $\varphi \in \widetilde{\mathbf{H}}_0$*. Or, in our specific problem, is equivalent to assume*

$$\frac{L}{\pi} C\sqrt{2} < \nu, \quad C \text{ given by (2.4.25) or (2.4.44).}$$

Remark 3.2.2 *We can prove also the uniqueness of weak solution by considering the forces field given by the simpler expression (2.2.11) and, consequently, with no need of the non-increasing assumption (2.4.56). In this case, we have from (3.2.20), with the forces field (2.2.11),*

$$\nu \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \; d\mathbf{x} + \delta \int_\Omega (|u_1|^{\sigma-1} u_1 - |u_2|^{\sigma-1} u_2, 0) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, d\mathbf{x} =$$

$$-\int_\Omega (\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{u}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, d\mathbf{x}.$$

Proceeding as in Remark 2.4.7 and using the assumption aforementioned in Remark 3.2.1, we prove that $\mathbf{u}_1 = \mathbf{u}_2$.

3.3 Localization Effects

In the previous section has been established the existence of a weak solution to problem (3.1.1)-(3.1.5) having a finite global energy (2.5.58).

Theorem 3.3.1 *Assume that* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). If* $\mathbf{u}$ *is any weak solution of the problem (3.1.1)-(3.1.5), with finite energy (2.5.58), then* $\mathbf{u}(x,y) = \mathbf{0}$ *for* $x > a'$*, where* $a' = a'(E, L, \delta, \nu, \sigma)$ *is a positive constant.*

The proof of this theorem, although to be much more technical, follows exactly the same reasoning of the proof of Theorem 2.5.1.

Definition 3.3.1 *A function ψ is a weak solution of problem (3.1.6)-(3.1.9), if:*
(i) $\psi \in \mathrm{H}^2(\Omega)$, $\mathbf{f}(\mathbf{x}, \psi_y, -\psi_x) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(ii) $\psi(0,y) = \int_0^y u_*(s)ds$, $\frac{\partial \psi}{\partial n}(0,y) = v_*(y)$, $\psi(x,0) = \psi(x,L) = \frac{\partial \psi}{\partial n}(x,0) = \frac{\partial \psi}{\partial n}(x,L) = \psi(0,L) = 0$, *and* ψ, $|\nabla\psi| \to 0$, *when* $x \to \infty$;
(iii) For every $\phi \in \mathrm{H}^2_0(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$ *with compact support,*

$$\nu \int_\Omega \triangle\psi \triangle\phi \, d\mathbf{x} - \int_\Omega (f_1 \phi_y - f_2 \phi_x) \, d\mathbf{x} = \int_\Omega \triangle\psi \, (\psi_x \phi_y - \psi_y \phi_x) \, d\mathbf{x}.$$

The following lemmas can be proved in the same manner as the correspondent ones stated in Section 2.5.

Lemma 3.3.1 *If* $\mathbf{u}$ *is a weak solution of (3.1.1)-(3.1.5) in the sense of Definition 3.2.1, then* ψ*, given by (2.3.15), is a weak solution of (3.1.6)-(3.1.9) in the sense of Definition 3.3.1.*

Lemma 3.3.2 *Let* ψ *be a weak solution of (3.1.6)-(3.1.9) with a finite global energy (2.5.58). Assume that* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). Then, for every* $a \geq x_g$ *and every positive integer* $m \geq 2$

$$\int_\Omega \left(\nu |\mathrm{D}^2\psi|^2 + \delta |\psi_y|^{1+\sigma}\right) (x-a)_+^m d\mathbf{x} \leq$$

$$2m\nu \int_\Omega |\triangle\psi| \, |\psi_x| (x-a)_+^{m-1} d\mathbf{x} + 2m\nu \int_\Omega |\psi_y| |\psi_{xy}| (x-a)_+^{m-1} d\mathbf{x} \tag{3.3.21}$$
$$+m(m-1)\nu \int_\Omega |\triangle\psi| \, |\psi| (x-a)_+^{m-2} d\mathbf{x} + m \int_\Omega |\triangle\psi| \, |\psi_y| |\psi| (x-a)_+^{m-1} d\mathbf{x}.$$

From the term on the left-hand side of (3.3.21), it will arise the energy type terms (2.5.63) depending on a and which will appear in the more difficult part of the proof of Theorem 3.3.1 expressed by the following lemma[2].

Lemma 3.3.3 *Let* ψ *be a weak solution of (3.1.6)-(3.1.9) and let us assume* $\mathbf{f}$ *satisfies (2.2.9) and (2.2.10). Then, the following differential inequality holds for* $a \geq x_g$ *(*x_g *is given in (2.2.10)):*

$$E_m(a) \leq C_1 \left(E_{m-2}(a)\right)^{\mu_1} + C_2 \left(E_{m-2}(a)\right)^{\mu_2}, \tag{3.3.22}$$

for every integer $m > 3$*, where* $C_i = C_i(L, m, \delta, \nu, \sigma)$*,* $i = 1, 2$ *are positive constants and* $\mu_j = \mu_j(m, \sigma) > 1$*,* $j = 1, 2$*. Moreover,* $E_2(a) < \infty$ *for any* $a \geq x_g$*. In fact,*

$$E_2(a) \leq C_3 E_0(a) + C_4 \left(E_0(a)\right)^{\mu_1} + C_5 \left(E_0(a)\right)^{\mu_2}, \tag{3.3.23}$$

where C_3*,* $C_i = C_i(L, \delta, \nu, \sigma)$*,* $i = 4, 5$*, are positive constants, with* C_3 *an universal one, and* $\mu_j = \mu_j(\sigma) > 1$*,* $j = 1, 2$*.*

[2] As in Section 2.5, we will numerate the constants appearing in the curse of this proof, just to simplify the exposition.

Proof: We rewrite (3.3.21) as

$$\int_\Omega \left(\nu|\mathrm{D}^2\psi|^2 + \delta|\psi_y|^{1+\sigma}\right)(x-a)_+^m d\mathbf{x} \leq 2m\nu I_1 + 2m\nu I_2 + m(m-1)\nu I_3 + mJ.$$

Applying Cauchy's inequality with $\varepsilon = \nu/(2m)$ to the term J and then taking the minimum on the left-hand side, we obtain

$$\min\left(\frac{\nu}{2},\delta\right) E_m(a) \leq 2m\nu I_1 + 2m\nu I_2 + m(m-1)\nu I_3 + \frac{m^2}{\nu} J_2,$$

where

$$J_2 = \int_\Omega \psi_y^2 \psi^2 (x-a)_+^{m-2}\, d\mathbf{x}.$$

If we assume $m > 3$, the estimations of I_1, I_2 and I_3 obtained in Section 2.5 lead to

$$\min\left(\frac{\nu}{2},\delta\right) E_m(a) \leq \varepsilon C_6 E_m(a) + \frac{1}{\varepsilon} C_7 \left(E_{m-2}(a)\right)^\mu + C_8 J_2, \tag{3.3.24}$$

where μ is given by (2.5.75), $C_6 = C_6(m,\nu)$, $C_7 = C_7(L,m,\nu,\sigma)$ and $C_8 = C_8(m,\nu)$. To estimate J_2, we use two fundamental one-dimensional inequalities. The first one is Poincaré's inequality (1.1.3), with $N = 1$, and the second is Ladyzhenskaya's inequality[3]

$$\int_0^L u(y)^4\, dy \leq 2\left(\int_0^L u(y)^2\, dy\right)\left(\int_0^L u'(y)^2\, dy\right), \tag{3.3.25}$$

valid for sufficiently regular functions u such that $u(0) = u(L) = 0$. Then, we apply Hölder's inequality to J_2 to obtain,

$$\int_0^L \phi^2\, dy = -2\int_x^\infty \int_0^L \phi\phi_x\, dy dx \leq 2\left(\int_\Omega \phi^2\, d\mathbf{x}\right)^{\frac{1}{2}} \left(\int_\Omega \phi_x^2\, d\mathbf{x}\right)^{\frac{1}{2}}$$

for every $x \geq 0$ and every function ϕ with the same regularity and boundary values of our ψ or ψ_y. In a final step, we use Cauchy's inequality, and from the definition of $E_0(0) \equiv E$ and $E_{m-2}(a)$, we prove

$$J_2 \leq C_9\, E\, \left(E_{m-2}(a)\right)^{\frac{\mu+1}{2}}, \quad C_9 = C_9(L,m,\nu,\sigma), \tag{3.3.26}$$

where θ is given by (2.5.73). Then, (3.3.24) becomes

$$\min\left(\frac{\nu}{2},\delta\right) E_m(a) \leq \varepsilon C_6 E_m(a) + \frac{1}{\varepsilon} C_7 \left(E_{m-2}(a)\right)^\mu + C_8 \left(E_{m-2}(a)\right)^{\frac{\mu+1}{2}},$$

where, now, $C_8 = C_8(E,L,m,\nu,\sigma)$. Then, choosing an appropriated ε, we obtain the fractional differential inequality (3.3.22).

[3]Inequalities of the form (3.3.25) have been widely used in investigation of uniqueness for the Navier-Stokes equations in two dimensions (see Ladyzhenskaya [72]). See Horgan [62] and the references therein for some results concerned with obtaining optimal constants.

If $m = 2$, the estimates on I_1, I_2 and I_3 (with $m = 2$) obtained in Section 2.5 lead to

$$\min\left(\frac{\nu}{2}, \delta\right) E_2(a) \leq \varepsilon C_{10} E_2(a) + \varepsilon C_{11} E_0(a) + \frac{1}{\varepsilon} C_{12} \left(E_0(a)\right)^{\mu} + C_{13} J_{2(m=2)},$$

where θ and μ (with $m = 2$) are given by (2.5.81), $C_{10} = C_{10}(\nu)$, $C_{11} = C_{11}(\nu)$ ($C_{11} = 2C_{10}$), $C_{12} = C_{12}(L, \nu, \sigma)$ and $C_{13} = C_{13}(\nu)$. Taking $m = 2$ in (3.3.26),

$$J_{2(m=2)} \leq C_{14}\, E\, \left(E_0(a)\right)^{\frac{\mu+1}{2}}, \quad C_{14} = C_9 \text{ with } m = 2.$$

Finally, choosing a suitable ε, we obtain the differential inequality (3.3.23).□

Proof: [Theorem 3.3.1]. Taking $m = 4$ in Lemma 3.3.3, we have the fractional differential inequality

$$E_4(a) \leq C_{15} \left(E_2(a)\right)^{\mu_1} + C_{16} \left(E_2(a)\right)^{\mu_2}, \tag{3.3.27}$$

where, from (2.5.75),

$$\mu_1 = \mu = \frac{5 - \sigma}{4} \quad \text{and} \quad \mu_2 = \frac{\mu + 1}{2} = \frac{9 - \sigma}{8}$$

and $C_{15} = C_{15}(L, m, \delta, \nu, \sigma)$, $C_{16} = C_{16}(L, m, \delta, \nu, \sigma)$. Using Lemma 3.3.3, with $m = 2$, and because of the finiteness of E, we can easily see that $E_2(a)$ is finite. To conclude the proof of Theorem 3.3.1, we need the following auxiliary result whose proof is given in Section 3.4.

Lemma 3.3.4 *Assume that the fractional differential inequality*

$$E_m(a) \leq C_{17} \left(E_{m-p}(a)\right)^{\mu_1} + C_{18} \left(E_{m-p}(a)\right)^{\mu_2} \tag{3.3.28}$$

holds for every $a \geq 0$, where $0 < p < m < w = p\gamma/(\gamma - 1)$, C_{17} and C_{18} are positive constants and $1 < \mu_1,\ \mu_2 < m/(m-p)$ and $\gamma = \min(\mu_1, \mu_2)$. Assume $E_{m-p}(a)$ is finite for every $a \geq 0$. Then, the support of $E_0(a)$ is a bounded interval $[0,\ a_*]$, with $a_* \leq a'$, where

$$a' = (w - m + 1) C_{19}^{\frac{1}{(w-m)(\gamma-1)}} E^{\frac{1}{w-m}}, \quad C_{19} = C_{19}(C_{17}, C_{18}, E, m, p, \mu_1, \mu_2). \tag{3.3.29}$$

Then, from Lemma 3.3.4 applied to (3.3.27), with $m = 4$, $p = 2$, $w = 2(9-\sigma)/(1-\sigma) > m$, $1 < \mu_1 = (5 - \sigma)/4 < m/(m - p)$ and $1 < \mu_2 = (9 - \sigma)/8$ $(\equiv \gamma)$ $< m/(m - p)$, because $0 < \sigma < 1$, the support of $E_0(a)$ is a bounded interval $[0, a^*]$, with $a^* \leq a'$, where from (3.3.29)

$$a' = \frac{15 + \sigma}{1 - \sigma} C_{20}^{\frac{4}{7+\sigma}} E^{\frac{1}{2(7+\sigma)}}, \quad C_{20} = C_{20}(E, L, \delta, \nu, \sigma).$$

Then $E_0(a) = 0$ for $a > a'$, which implies $u = 0$ and v is constant almost everywhere for $x > a'$. Finally from (3.1.4), $v = 0$ too in the same domain.□

Remark 3.3.1 *Again, the localization effect can be extended to the limit case of $\sigma = 0$, but, in the case of $\sigma = 1$, the above arguments lead to the exponential decay (2.5.86). In Horgan [62] is obtained an analogous exponential decay using the same techniques of Toupin [117] and Knowles [68] referred in Remark 2.5.4. There it was considered problem (3.1.1)-(3.1.5) with $\mathbf{f} = \mathbf{0}$ under the assumption that, as x tends to infinity, $\mathbf{u}$ tends to the fully developed Poiseuille flow corresponding to the net inflow $\int_0^L u_*(y)\,dy = Q$.*

Remark 3.3.2 *A simple proof of the localization effect can be obtained by proving that*

$$|\psi| \leq C\|\psi\|_{\mathrm{H}^2(\Omega)}, \quad C = C(L), \tag{3.3.30}$$

and (3.3.22) and (3.3.23) would come as in Section 2.5 (see (2.5.68) and (2.5.69)), but with the constants depending also on $\|\psi\|_{\mathrm{H}^2(\Omega)}$. Another idea of a simple proof, is to assume $E_m(a) \leq 1$ or $E_m(a) \geq 1$ and again (3.3.22) and (3.3.23) would come as in Section 2.5, with $\mu = \min(\mu_1, \mu_2)$ or $\mu = \max(\mu_1, \mu_2)$, respectively.

Remark 3.3.3 *One can easily verify that Remarks 2.5.2 and 2.5.3 are valid for the stationary Navier-Stokes problem. Moreover, we can carry out, for this case, the result establishing the stagnation effect proved in Section 2.6, as well the results with localized forces field stated in Section 2.7 (cf. Antontsev et al [9]).*

3.4 Appendix

Here we prove Lemma 3.3.4 whose applications go beyond this thesis. Bernis [21] has proved a result, of which Lemma 2.5.5 is a direct consequence, for an ordinary differential inequality like (2.5.66). Latter, Bernis [19] extended this result for an ordinary differential inequality like

$$(E_m(a))^{\mu_0} \leq C \sum_{i=1}^{k} (E_{m-i}(a))^{\mu_i}, \quad k \leq m,$$

where $C, \mu_0, \ldots, \mu_k$ are positive constants, with $\mu_0, \ldots, \mu_k$ obeying to specific relations. But none of these results can be directly applied to Lemma 3.3.4, because $E_{m-i}(a)$ on the right-hand side of (3.3.27) is fixed with $m = 4$ and $i = 2$.

Proof: [Lemma 3.3.4]. For the sake of simplicity, let us put $C_{17} = C_1$, $C_{18} = C_2$ and $C_{19} = C$. For all $0 < p < m$ and all $a \geq 0$, we have by Hölder's inequality

$$E_{m-p}(a) \leq (E_m(a))^{\frac{m-p}{m}} (E_0(a))^{\frac{p}{m}}. \tag{3.4.31}$$

Given $\gamma = \min(\mu_1, \mu_2)$, by the monotonicity of $E_{m-p}(a)$, we have for $i = 1, 2$,

$$(E_{m-p}(a))^{\mu_i} \leq (E_{m-p}(0))^{\mu_i - \gamma} (E_{m-p}(a))^{\gamma}$$

and, from (3.4.31), we have for $i = 1, 2$,

$$(E_{m-p}(a))^{\mu_i} \leq (E_m(0))^{\frac{m-p}{m}(\mu_i-\gamma)} (E_0(0))^{\frac{p}{m}(\mu_i-\gamma)} (E_{m-p}(a))^{\gamma}. \tag{3.4.32}$$

On the other hand, from (3.3.28) and (3.4.31),

$$E_m(0) \leq \sum_{i=1}^{2} C_i \left(E_m(0)\right)^{\frac{m-p}{m}\mu_i} \left(E_0(0)\right)^{\frac{p}{m}\mu_i}.$$

Requiring that $\mu_i < m/(m-p)$, for $i = 1, 2$, we obtain, when using Young's inequality with $\varepsilon = 1/[2(C_1 + C_2)]$, $E_m(0) \leq H$, where

$$H := \sum_{i=1}^{2} 2C_i \left[2(C_1 + C_2)\frac{m-p}{m}\mu_i\right]^{\frac{(m-p)\mu_i}{m-(m-p)\mu_i}} \frac{m-(m-p)\mu_i}{m} \left(E_0(0)\right)^{\frac{p\mu_i}{m-(m-p)\mu_i}}.$$

Then, for $i = 1, 2$, (3.4.32) becomes

$$\left(E_{m-p}(a)\right)^{\mu_i} \leq H^{\frac{m-p}{m}(\mu_i-\gamma)} \left(E_0(0)\right)^{\frac{p}{m}(\mu_i-\gamma)} \left(E_{m-p}(a)\right)^{\gamma}. \tag{3.4.33}$$

From (3.3.28), (3.4.31) and (3.4.33),

$$E_m(a) \leq \sum_{i=1}^{2} C_i H^{\frac{m-p}{m}(\mu_i-\gamma)} \left(E_0(0)\right)^{\frac{p}{m}(\mu_i-\gamma)} \left(E_m(a)\right)^{\frac{m-p}{m}\gamma} \left(E_0(a)\right)^{\frac{p}{m}\gamma}.$$

Then, requiring that $\gamma < m/(m-p)$ and $w = p\gamma/(\gamma - 1) > m$,

$$E_m(a) \leq C^{\frac{m}{(w-m)(\gamma-1)}} \left(E_0(a)\right)^{\frac{p\gamma}{(w-m)(\gamma-1)}}, \tag{3.4.34}$$

where

$$C = \sum_{i=1}^{2} C_i \left\{2C_i \left[2(C_1 + C_2)\frac{m-p}{m}\mu_i\right]^{\alpha_i} \frac{m-(m-p)\mu_i}{m} E^{\beta_i}\right\}^{\gamma_i} E^{\delta_i},$$

with $\alpha_i = (m-p)\mu_i/\left[m-(m-p)\mu_i\right]$, $\beta_i = p\mu_i/\left[m-(m-p)\mu_i\right]$, $\gamma_i = (m-p)(\mu_i - \gamma)/m$, $\delta_i = \left[p(\mu_i - \gamma)\right]/m$. Let us put $m - 1 = p$ in (3.4.31). Then, from (3.4.34),

$$E_1(a) \leq C^{\frac{1}{(w-m)(\gamma-1)}} \left(E_0(a)\right)^{\frac{w-m+1}{w-m}}.$$

Since $E_1' = -E_0$, this is a first order differential inequality, whose explicit integration ends the proof.□

Remark 3.4.1 *The result proved above can be generalized for any energy function*

$$E_m(a) = \int_a^{\infty} f(x)(x-a)^m \, dx,$$

where $f \in \mathrm{L}^1(0,\infty)$ is such that $f \geq 0$ almost everywhere in $(0,\infty)$.

Chapter 4

Stationary Boussinesq Problem

"Indeed, many problems in thermal convection would be analytically intractable without the use of Boussinesq approximations."

J.M. Mihaljan [86, p. 1126].

A non-standard Boussinesq problem is considered in this chapter. This problem involves a system, in a semi-infinite strip, given by the stationary Navier-Stokes equations coupled with a stationary, and possibly nonlinear, advection diffusion equation for the temperature. We start by deriving the Boussinesq approximation in Section 4.1 where, besides the cited bibliography, we also have used the monograph by Kundu [71] and the article by Mihaljan [86]. The problem is presented in Section 4.2, where the term non-standard is used in the sense that Navier-Stokes equations are perturbed with a sublinear term $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u})$. We prove some existence and uniqueness results, in Section 4.3, using the same ideas of the foregoing chapters. In Section 4.4 we prove that the velocity vanishes for x large enough and in Section 4.5 we establish this localization effect for the same Boussinesq problem but in the case when the viscosity depends on the temperature. In consequence of this localization effect for the velocity, in Section 4.6, is proved the temperature has exponential decay.

4.1 Introduction

When we deal with flow of a fluid driven by *buoyancy forces*[1], as those caused by heating the fluid, equations (1.4.11) and (1.4.19) are not sufficient to describe the flow and it is also necessary to add an equation for the temperature θ:

$$\frac{\partial\,(\rho\, c\,\theta)}{\partial t} + \nabla(\rho\, c\,\theta)\cdot\mathbf{u} = -p\,\mathrm{div}\mathbf{u} + \lambda(\mathrm{div}\mathbf{u})^2 + 2\mu\,|\mathbf{D}|^2 + \mathrm{div}\,(k\nabla\theta)\,. \tag{4.1.1}$$

This equation is derived from the energy equation (1.4.20) by assuming

$$e = c\,\theta, \tag{4.1.2}$$

where $c \geq 0$ is a scalar function called *specific heat*, which in most cases, is taken to be a function of ρ and θ or simply a constant called the *specific heat coefficient*[2]. We must notice that the introduced constant c in (4.1.2) it is implied to mean the specific heat at constant volume (c_V) and not the specific heat at constant pressure (c_P) which appears in the *enthalpy*[3] relation $i = c\,\theta$ (see *e.g.* Batchelor [16]). For many convective flows the system $\{(1.4.11), (1.4.19), (4.1.1)\}$ may be considerably simplified by assuming the fluid is *isochoric*, *i.e.* essentially incompressible, except in the body force term $\mathbf{f}$ of (1.4.19). This approximation is known in the literature as the *Boussinesq approximation*[4] (see *e.g.* Straughan [112]) and allow us to avoid the difficult problem associated with the compressible Navier-Stokes equations by taking advantage of simplifying the features which characterize the fluid. These are:

(1) The fluid is as if incompressible except that density is not ignored in the body force term of the motion equation (1.4.19);

(2) The density changes are induced by changes of temperature and concentration, but not by pressure;

(3) The velocity gradients are sufficiently small so that the effect on the temperature of conversion of work to heat can be ignored.

In the Boussinesq approximation of a large class of flow problems, thermodynamics parameters such as viscosity, thermal conductivity and specific heat can be assumed constant leading to a coupled system with linear second order operators in the Navier-Stokes equations (1.4.23) and temperature equation (4.1.1) (see *e.g.* Joseph [64] and Straughan [112]). However there are some fluids like lubricants or some plasma flow for which this is no longer an accurate assumption (see *e.g.* Cannon et al [30] and Rodrigues [98]). In this situation, the Boussinesq approximation lead to the following system formed by (1.4.23), where $\mathbf{f} = \mathbf{f}(\theta)$ is the buoyancy force[5], (1.4.22) and the new equation for temperature

$$\frac{\partial\mathcal{C}(\theta)}{\partial t} + \mathbf{u}\cdot\nabla\mathcal{C}(\theta) = \triangle\varphi(\theta), \tag{4.1.3}$$

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[1] The upward forces acting on an object placed in a fluid.

[2] Generalized relations like $e = e(\theta)$ are known in the literature as (see P-L Lions [82]).

[3] A thermodynamic property of a system.

[4] The approximations generally attributed to Boussinesq (1903) were actually of earlier origin and were used by Oberbeck (1891) (see Joseph [64]).

[5] In fact, $\mathbf{f} = f(\theta)\mathbf{g}$, where $\mathbf{g}$ is a body force field, typically the gravity (see Joseph [64]).

where

$$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} c(s)\, ds \quad \text{and} \quad \varphi(\theta) := \int_{\theta_0}^{\theta} k(s)\, ds, \tag{4.1.4}$$

with c the specific heat and k the thermal conductivity.

4.2 Statement of the Problem

We shall consider a stationary Boussinesq coupling among the temperature θ and the velocity $\mathbf{u}$

$$-\nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p, \tag{4.2.5}$$

$$\operatorname{div}\mathbf{u} = \mathbf{0}, \tag{4.2.6}$$

$$\mathbf{u}\cdot\nabla\mathcal{C}(\theta) = \triangle\varphi(\theta), \tag{4.2.7}$$

in the semi-infinite strip $\Omega = (0,\infty)\times(0,L)$, $L>0$, where $\mathcal{C}$ and φ are given by (4.1.4) and ν is the kinematic viscosity coefficient. Motivated by the foregoing chapters, we assume the body force field is given in a nonlinear feedback form, $\mathbf{f}:\Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$, $\mathbf{f} = (f_1(\mathbf{x},\theta,\mathbf{u}), f_2(\mathbf{x},\theta,\mathbf{u}))$, such that, for every $\mathbf{u}\in\mathbb{R}^2$, $\mathbf{u}=(u,v)$, $\theta\in[m,M]$, with $m<M$ constants, and almost all $\mathbf{x}\in\Omega$

$$-\mathbf{f}(\mathbf{x},\theta,\mathbf{u})\cdot\mathbf{u} \geq \delta\,|u|^{1+\sigma(\theta)} - g(\mathbf{x},\theta) \tag{4.2.8}$$

for some $\delta>0$, σ a Lipschitz continuous function such that

$$0 < \sigma^- \leq \sigma(\theta) \leq \sigma^+ < 1 \quad \text{for every } \theta\in[m,M] \tag{4.2.9}$$

and

$$g\in \mathrm{L}^1\left(\Omega^{x_g}\times\mathbb{R}\right), \quad g\geq 0,\ g(\mathbf{x},\theta)=0 \text{ a.e. in } \Omega^{x_g} \text{ for every } \theta\in[m,M], \tag{4.2.10}$$

for some x_g, with $0\leq x_g<\infty$, and where $\Omega^{x_g}=(0,x_g)\times(0,L)$, $\Omega_{x_g}=(x_g,\infty)\times(0,L)$. We consider in the aforementioned domain Ω a planar stationary thermal flow of a fluid governed by (4.2.5)-(4.2.7) with the forces field $\mathbf{f}=\mathbf{f}(\mathbf{x},\theta,\mathbf{u})$ satisfying (4.2.8)-(4.2.10). Assuming $k>0$, then φ is invertible and so $\theta=\varphi^{-1}(\overline{\theta})$ for some real argument $\overline{\theta}$. Then we can define functions

$$\overline{\mathcal{C}}(\overline{\theta}) := \mathcal{C}\circ\varphi^{-1}(\overline{\theta}), \quad \overline{\mathbf{f}}(\mathbf{x},\overline{\theta},\mathbf{u}) := \mathbf{f}\circ\varphi^{-1}(\overline{\theta}), \quad \overline{\mu}(\overline{\theta}) := \mu\circ\varphi^{-1}(\overline{\theta}).$$

We point out that functions $\overline{\mathcal{C}}$, $\overline{\mathbf{f}}$ and $\overline{\mu}$ are Lipschitz continuous functions of $\overline{\theta}$. Substituting these expressions in (4.2.5)-(4.2.7), with $\mathbf{f}=\mathbf{f}(\mathbf{x},\theta,\mathbf{u})$, we get, omitting the bars,

$$-\nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\mathbf{x},\theta,\mathbf{u}) - \nabla p, \tag{4.2.11}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{4.2.12}$$

$$\mathbf{u}\cdot\nabla\mathcal{C}(\theta) = \triangle\theta. \tag{4.2.13}$$

To these equations we add the boundary conditions on $\mathbf{u}$ and the velocity at infinity

$$\mathbf{u}(0,y) = \mathbf{u}_*(y) \quad \text{for } y\in(0,L), \tag{4.2.14}$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0} \quad \text{for } x \in (0,\infty), \tag{4.2.15}$$

$$|\mathbf{u}(x,y)| \to 0 \quad \text{as } x \to \infty \text{ and } \; y \in (0,L). \tag{4.2.16}$$

We add also the boundary conditions on θ and the temperature at infinity

$$\theta = \theta_* \quad \text{on } \; x = 0, \; y = 0, \; L, \tag{4.2.17}$$

$$\theta(x,y) \to 0 \quad \text{as } x \to \infty \text{ and } \; y \in (0,L). \tag{4.2.18}$$

Here, $\mathbf{u}_*$ and θ_* are given functions with a suitable regularity to be indicated later on and

$$0 \le m \le \theta_*(\mathbf{x}) \le M < \infty. \tag{4.2.19}$$

We assume the possible non-zero velocity $\mathbf{u}_*$ and temperature θ_* satisfy the compatibility conditions (2.2.6)-(2.2.7) and

$$\theta_*(x,y) \to 0, \quad \text{as } x \to \infty \text{ and } y = 0, \; L. \tag{4.2.20}$$

4.3 Weak Formulation

We shall search solutions $(\theta, \mathbf{u})$ such that, additionally to assumptions (4.2.18) and (4.2.16), satisfy

$$\int_\Omega |\nabla \theta|^2 d\mathbf{x} < \infty \quad \text{and} \quad \int_\Omega |\nabla \mathbf{u}|^2 d\mathbf{x} < \infty.$$

Moreover, due to the fact that Poincaré's inequality (1.1.3) holds, our searched solution $(\theta, \mathbf{u})$ will be an element of the Sobolev product space $\mathrm{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$.

In order to define a weak solution for this problem, we recover the solenoidal vector spaces $\widetilde{\mathbf{H}}(\Omega)$ and $\widetilde{\mathbf{H}}_0(\Omega)$ defined in Section 2.4 and let us still denote by $\mathbf{u}_*$ and θ_* the extensions of the boundary data to the whole domain Ω in a way such that

$$\mathbf{u}_* \in \widetilde{\mathbf{H}}(\Omega) \quad \text{and} \quad \theta_* \in \mathrm{H}^1(\Omega) \cap \mathrm{C}^\alpha(\overline{\Omega}) \quad \text{for some } \alpha > 0. \tag{4.3.21}$$

Definition 4.3.1 *The pair $(\theta, \mathbf{u})$ is said to be a weak solution of (4.2.11)-(4.2.18) if: (i) $\theta - \theta_* \in \mathrm{H}^1(\Omega) \cap \mathrm{C}^\alpha(\overline{\Omega})$ for some $\alpha > 0$, $m \le \theta \le M$ and for every test function $\zeta \in \mathrm{H}^1_0(\Omega)$*

$$\int_\Omega (\nabla \theta - \mathcal{C}(\theta)\mathbf{u}) \cdot \nabla \zeta \, d\mathbf{x} = 0.$$

(ii) $\mathbf{u} \in \widetilde{\mathbf{H}}(\Omega)$, $\mathbf{u} - \mathbf{u}_* \in \widetilde{\mathbf{H}}_0(\Omega)$, $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u}) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$ and for every $\varphi \in \widetilde{\mathbf{H}}_0(\Omega) \cap \mathbf{L}^\infty(\Omega)$ with compact support,

$$\nu \int_\Omega \nabla \mathbf{u} : \nabla \varphi \, d\mathbf{x} + \int_\Omega \mathbf{u} \cdot \nabla \mathbf{u} \cdot \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x}. \tag{4.3.22}$$

In this section, we shall assume that $\mathbf{f} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by the structural condition

$$\mathbf{f}(\mathbf{x}, \theta, \mathbf{u}) = -\delta(|u|^{\sigma(\theta)-1}u, 0) - \mathbf{h}(\mathbf{x}, \theta, \mathbf{u}), \qquad (4.3.23)$$

for every $\mathbf{u} = (u, v)$, $\theta \in [m, M]$ and almost all $\mathbf{x} \in \Omega$, for some $\delta > 0$, and $0 < \sigma(\theta) < 1$ (recall (4.2.9)). Here, $\mathbf{h}(\mathbf{x}, \theta, \mathbf{u})$ is a Carathéodory function[6] such that

$$\mathbf{h}(\mathbf{x}, \theta, \mathbf{u}) \cdot \mathbf{u} \geq -g(\mathbf{x}, \theta), \qquad (4.3.24)$$

for every $\mathbf{u} \in \mathbb{R}^2$, $\theta \in [m, M]$ and almost all $\mathbf{x} \in \Omega$, for some function g satisfying (4.2.10). Furthermore, we assume

$$H_K \in \mathrm{L}^1(\Omega) \text{ for all } K > 0, \quad H_K(\mathbf{x}) = \sup_{|\mathbf{u}| \leq K,\ \theta \in [m,M]} |\mathbf{h}(\mathbf{x}, \theta, \mathbf{u})| \qquad (4.3.25)$$

and one of the following conditions holds:

(i) there exist some positive constants M, C, a function $G \in \mathrm{L}^p(\Omega \times \mathbb{R})$, for some $p > 1$ and $s \in (0, 1)$, such that

$$|\mathbf{h}(\mathbf{x}, \theta, \mathbf{u})| \leq C\,|\mathbf{u}|^s + G(\mathbf{x}, \theta), \qquad (4.3.26)$$

for every $\mathbf{u} \in \mathbb{R}^2$, $\theta \in [m, M]$ and almost all $\mathbf{x} \in \Omega$;

(ii) or there exists $\varepsilon > 0$ such that

$$|\angle(\mathbf{h}(\mathbf{x}, \theta, \mathbf{u}), \mathbf{u})| \notin \left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right) \qquad (4.3.27)$$

for every $|\mathbf{u}| > K$, $\theta \in [m, M]$ and almost all $\mathbf{x} \in \Omega$.

Theorem 4.3.1 *Let us assume $\mathbf{u}_* \in \mathbf{H}^{\frac{1}{2}}(0, L)$, $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u})$ satisfies (4.3.23)-(4.3.25) and assume that either growth condition (4.3.26) or vectors angle condition (4.3.27) holds. Then, under conditions[7] (4.2.9), (4.2.19)-(4.2.20) and (4.3.21), the problem (4.2.11)-(4.2.18) has, at least, one weak solution $(\theta, \mathbf{u})$.*

Proof: We will prove this theorem into three steps.
First step: an auxiliary problem for the temperature θ. Let

$$\mathbf{w} \in \mathbf{L}^2(\Omega) \cap \mathbf{L}^p(\Omega), \quad \text{with } N = 2 < p < \infty, \qquad (4.3.28)$$

be a given weakly divergence free vector function[8] and let us consider the following problem for the temperature

$$\mathbf{w} \cdot \nabla \mathcal{C}(\theta) = \triangle \theta \quad \text{in } \Omega, \qquad (4.3.29)$$

$$\theta = \theta_* \quad \text{on } \ x = 0,\ y = 0,\ L, \qquad (4.3.30)$$

---

[6] $\mathbf{f}(\mathbf{x}, \theta, \mathbf{u})$ is measurable in $\mathbf{x}$ for every $(\theta, \mathbf{u}) \in \mathbb{R} \times \mathbb{R}^2$ and continuous in $(\theta, \mathbf{u})$ for almost all $\mathbf{x} \in \Omega$.

[7] We assume also the hypothesis which guarantee the uniqueness of weak solutions in the Navier-Stokes problem.

[8] This means, $\int_\Omega \mathbf{w} \cdot \nabla \zeta \, d\mathbf{x} = 0$, for all $\zeta \in \mathrm{H}^1_0(\Omega)$ (see Cannon et al [29] and Rodrigues [98])

$$\theta(x,y) \to 0 \quad \text{as } x \to \infty \text{ and } y \in (0,L). \tag{4.3.31}$$

Since $\mathcal{C}$ is Lipschitz continuous we know[9] that problem (4.3.29)-(4.3.31), assuming (4.3.28), has a unique weak solution $\theta \in \mathrm{H}^1(\Omega) \cap \mathrm{C}^\alpha(\overline{\Omega})$, for some $\alpha : 0 < \alpha < 1$, such that

$$\|\theta\|_{\mathrm{H}^1(\Omega)}\,,\ \|\theta\|_{\mathrm{C}^\alpha(\overline{\Omega})} \leq C\left(p, \|\mathbf{w}\|_{\mathbf{L}^p(\Omega)}\,, \|\theta_*\|_{\mathrm{H}^1(\Omega)}\right). \tag{4.3.32}$$

Moreover, from the Maximum Principle, $m \leq \theta(x) \leq M$. Then we can define the nonlinear operator

$$\Lambda : \mathbf{L}^2(\Omega) \cap \mathbf{L}^p(\Omega) \to \mathrm{H}^1(\Omega) \cap \mathrm{C}^\alpha(\overline{\Omega}), \quad \Lambda(\mathbf{w}) = \theta, \tag{4.3.33}$$

for some $\alpha > 0$ and $N = 2 < p < \infty$. The operator Λ is continuous, because from (4.3.32), we get that given a sequence $\mathbf{w}_n$ such that

$$\|\mathbf{w}_n - \mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{w}_n - \mathbf{w}\|_{\mathbf{L}^p(\Omega)} \to 0 \quad \text{as } n \to \infty,$$

then

$$\|\Lambda(\mathbf{w}_n) - \Lambda(\mathbf{w})\|_{\mathrm{H}^1(\Omega)} + \|\Lambda(\mathbf{w}_n) - \Lambda(\mathbf{w})\|_{\mathrm{C}^\alpha(\overline{\Omega})} \to 0, \quad \text{as } n \to \infty.$$

Second step: an auxiliary problem for the velocity $\mathbf{u}$. Let ω be a given function such that

$$\omega \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^\alpha(\overline{\Omega}), \quad 2 < p < \infty, \quad \alpha > 0, \quad m \leq \omega \leq M \tag{4.3.34}$$

and let us consider the following problem for the velocity

$$-\nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}(\mathbf{x}, \omega, \mathbf{u}) - \nabla p \quad \text{in } \Omega, \tag{4.3.35}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{4.3.36}$$

$$\mathbf{u}(0,y) = \mathbf{u}_*(y) \quad \text{for } y \in (0,L), \tag{4.3.37}$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0} \quad \text{for } x \in (0,\infty), \tag{4.3.38}$$

$$|\mathbf{u}(x,y)| \to 0 \quad \text{as } x \to \infty \text{ and } \ y \in (0,L). \tag{4.3.39}$$

Applying the results of Section 3.2, which is possible due to the assumptions (4.3.23)-(4.3.25) and (4.3.26) or (4.3.27), problem (4.3.35)-(4.3.39), with a given ω satisfying (4.3.34), has a unique weak solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$ which satisfies

$$\int_\Omega \left(|\nabla \mathbf{u}|^2 + |u|^{1+\sigma(\omega)} + |\mathbf{h}(\mathbf{x}, \omega, \mathbf{u}) \cdot \mathbf{u}|\right) d\mathbf{x} \leq C, \tag{4.3.40}$$

where, if we are using (4.3.26) (see (2.4.25)),

$$C = C\left(L, m, M, \delta, s, p, \nu, \sigma, \|g\|_{\mathrm{L}^1(\Omega^{x_g} \times \mathbb{R})}, \|G\|_{\mathrm{L}^p(\Omega \times \mathbb{R})}, \|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right),$$

[9] For this concrete problem, see Cannon et al [29] and Rodrigues [98] and for a general elliptic boundary value problem see Carrillo and Chipot [31] and the fundamental reference Ladyzhenskaya and Ural'tseva [74, pp. 70-95, 199-203].

or, if we are using (4.3.27) (see (2.4.44)),

$$C = C\left(L, m, M, \delta, \nu, \sigma, \|g\|_{\mathrm{L}^1(\Omega^{x_g}\times\mathbb{R})}, \|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)}\right).$$

Then, we can define the nonlinear operator

$$\Pi : \mathrm{H}^1(\Omega)\cap \mathrm{C}^{\alpha}(\overline{\Omega}) \to \mathbf{L}^p(\Omega), \quad \Pi(\omega) = \mathbf{u}, \tag{4.3.41}$$

for some $\alpha > 0$ and $N = 2 < p < \infty$, which is continuous.

Third step: application of Schauder's theorem. Given $p > 2$, formulas (4.3.33) and (4.3.41) allow us to define the composition nonlinear operator

$$\Upsilon = \Pi\Lambda : \mathbf{L}^2(\Omega)\cap\mathbf{L}^p(\Omega) \to \mathbf{L}^p(\Omega). \tag{4.3.42}$$

From (4.3.40) we get that Υ transforms $\mathbf{L}^2(\Omega)\cap\mathbf{L}^p(\Omega)$ into a bounded subset of $\mathbf{H}^1(\Omega)$ and, from the Sobolev compact imbedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^p(\Omega)$, $2 < p < \infty$, it is completely continuous. Then, according to Schauder's theorem, (4.3.42) has, at least, a fixed point. This proves the existence of a weak solution $(\theta, \mathbf{u})$ to problem (4.2.11)-(4.2.18).□

To establish an uniqueness result like Theorems 2.4.3 and 3.2.3, we consider a non-increasing condition analogous to (2.4.56),

$$(\mathbf{f}(\mathbf{x}, \theta_1, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \theta_2, \mathbf{u}_2))\cdot(\mathbf{u}_1 - \mathbf{u}_2) \le 0 \tag{4.3.43}$$

for every $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{R}^2$, θ_1, $\theta_2 \in [m, M]$ and almost all $\mathbf{x} \in \Omega$. Again, because of the nonlinearity of the Navier-Stokes equations, we need to assume that ν and $\mathbf{u}_*$ are such that the problem (4.3.35)-(4.3.39), with $\mathbf{f}(\mathbf{x}, \omega, \mathbf{0})$ and ω given by (4.3.34), has a unique weak solution in $\mathbf{H}^1(\Omega)$. In this case, we need an extra condition, on the function $\mathcal{C}$, to guarantee the uniqueness result.

Theorem 4.3.2 *Assume (4.2.9), (4.2.19)-(4.2.20) and (4.3.43). We additionally assume that*

$$|\mathcal{C}'(\theta)| \le \lambda \quad \textit{for every } \theta \in [m, M], \tag{4.3.44}$$

for some small enough positive constant λ. Then, if $\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)} \le \varepsilon^*\nu$ for some small enough positive constant $\varepsilon^* > 0$, the problem (4.2.11)-(4.2.18) has a unique weak solution $(\theta, \mathbf{u})$.

Proof: Let $(\theta_1, \mathbf{u}_1)$, $\mathbf{u}_1 = (u_1, v_1)$, and $(\theta_2, \mathbf{u}_2)$, $\mathbf{u}_2 = (u_2, v_2)$, be two weak solutions of problem (4.2.11)-(4.2.18) and let us set $\theta = \theta_1 - \theta_2$, $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. According to Definition 4.3.1, $\mathbf{u} = \mathbf{u_1} - \mathbf{u_2} \in \mathbf{H}^1_0(\Omega)$, $\theta = \theta_1 - \theta_2 \in \mathrm{H}^1_0(\Omega)$ and setting $\zeta = \theta$ and $\varphi = \mathbf{u}$, we come to the relations

$$\int_\Omega |\nabla\theta|^2 d\mathbf{x} = \int_\Omega (\mathcal{C}(\theta_1) - \mathcal{C}(\theta_2))\,\mathbf{u}_1\cdot\nabla\theta\, d\mathbf{x} + \int_\Omega \mathcal{C}(\theta_2)\mathbf{u}\cdot\nabla\theta\, d\mathbf{x} := T_1 + T_2 \tag{4.3.45}$$

and

$$\nu\int_\Omega |\nabla\mathbf{u}|^2\, d\mathbf{x} + \int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}_1\cdot\mathbf{u}\, d\mathbf{x} = \int_\Omega (\mathbf{f}(\mathbf{x}, \theta_1, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \theta_2, \mathbf{u}_2))\cdot\mathbf{u}\, d\mathbf{x}. \tag{4.3.46}$$

Estimate for the temperature. Using (4.3.44) and Cauchy's inequality, we get

$$|T_1| \leq \lambda \int_\Omega |\theta|\,|\mathbf{u}_1|\,|\nabla\theta| d\mathbf{x} \leq \frac{1}{4}\int_\Omega |\nabla\theta|^2\, d\mathbf{x} + \lambda^2 \int_\Omega |\theta|^2 |\mathbf{u}_1|^2\, d\mathbf{x} \tag{4.3.47}$$

and

$$|T_2| \leq C \int_\Omega |\mathbf{u}|\,|\nabla\theta|\, d\mathbf{x} \leq \frac{1}{4}\int_\Omega |\nabla\theta|^2\, d\mathbf{x} + C^2 \int_\Omega |\mathbf{u}|^2\, d\mathbf{x}, \tag{4.3.48}$$

with $C = C(m,M) = \max_{m\leq\theta\leq M} |\mathcal{C}(\theta)|$. We use Poincaré's inequalities

$$\int_0^L |\mathbf{u}|^2\, dy \leq C \int_\Omega |\nabla \mathbf{u}|^2\, d\mathbf{x} \quad \text{and} \quad |\theta(x,y)|^2 \leq C \int_0^L |\theta_y(x,s)|^2\, ds,$$

where $C = C(L)$ are two different positive constants, to obtain, from (4.3.40), that

$$\int_\Omega |\theta|^2 |\mathbf{u}_1|^2\, d\mathbf{x} \leq C \int_\Omega |\nabla\theta|^2\, d\mathbf{x}, \tag{4.3.49}$$

where C is a positive constant with the same dependence as in (4.3.40). Joining (4.3.45), (4.3.47), (4.3.48) and (4.3.49), we arrive to

$$\frac{1}{2}\int_\Omega |\nabla\theta|^2\, d\mathbf{x} \leq C\lambda^2 \int_\Omega |\nabla\theta|^2\, d\mathbf{x} + C\int_\Omega^L |\nabla\mathbf{u}|^2\, d\mathbf{x},$$

with C given in (4.3.49). Choosing λ such that $2C\lambda^2 < 1$, it results

$$\int_\Omega |\nabla\theta|^2\, d\mathbf{x} \leq C \int_\Omega |\nabla\mathbf{u}|^2\, d\mathbf{x}. \tag{4.3.50}$$

Estimate for the velocity. Arguing as for the proof of Theorem 3.2.3, we obtain from (4.3.46)

$$\nu \int_\Omega |\nabla\mathbf{u}|^2\ \ d\mathbf{x} \leq -\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}_1\cdot\mathbf{u}\, d\mathbf{x} := U. \tag{4.3.51}$$

By using Hölder's inequality, the Sobolev imbedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ and (4.3.40), we can estimate U in the following way

$$|U| \leq \|\nabla\mathbf{u}_1\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|^2_{\mathbf{L}^4(\Omega)} \leq C\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)} \|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}, \tag{4.3.52}$$

where C is given in (4.3.40). Thus, by assuming that $C\|\mathbf{u}_*\|_{\mathbf{H}^{\frac{1}{2}}(0,L)} < \nu$, using (4.3.52) and Poincaré's inequality, we obtain, from (4.3.51), that $\mathbf{u}_1 = \mathbf{u}_2$ and, as consequence of (4.3.50) and Poincaré's inequality, $\theta_1 = \theta_2$. □

Remark 4.3.1 *Condition (4.3.44) may be replaced by the condition*

$$M - m = \lambda$$

for some positive small enough λ, where m and M are given in (4.2.19). Here $\mathcal{C} \in$ $\mathrm{C}^2(m,M)$ and according to Lagrange's theorem, $\mathcal{C}' = \mathcal{C}''\theta$. Then

$$|\mathcal{C}'| \leq \max_{\theta\in[m,M]} |\mathcal{C}''|\,|M-m| \leq C\lambda.$$

Remark 4.3.2 *Analogously to Remarks 2.4.7 and 3.2.2, there is a situation for which we can prove the uniqueness of solution for problem (4.2.11)-(4.2.18) by dropping the non-increasing condition (4.3.43). Here, the body forces field is assumed to be given by*

$$\mathbf{f}(\mathbf{x},\theta,\mathbf{u}) = -\delta(|u|^{\sigma(\theta)-1}u, 0).$$

In this case, besides (4.3.44), we need to assume furthermore

$$|\sigma'(\theta)| \leq \lambda \quad \textit{for every } \theta \in [m, M].$$

These proof is a little bit more involving than those in Remarks 2.4.7 and 3.2.2 and we address the reader to Antontsev et al [10].

4.4 Localization Effects

In this section we study the localization effect for the velocity $\mathbf{u}$ associated to the problem (4.2.11)-(4.2.18). It turns out that the qualitative property of the spatial localization of $\mathbf{u}$ is independent of the temperature component θ. So, if we are not interested to know how big is the support of $\mathbf{u}$, but merely in knowing that support of $\mathbf{u}$ is a compact subset of Ω we can assume θ be given. In this way, our problem becomes simpler than before, since there is none partial differential equation for θ. Thus, given θ such that

$$\theta \in \mathrm{L}^\infty(\Omega), \quad \theta(\mathbf{x}) \in [m, M] \text{ for almost all } \mathbf{x} \in \Omega, \tag{4.4.53}$$

we consider the following auxiliary problem

$$-\nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\mathbf{x},\theta,\mathbf{u}) - \nabla p \quad \text{in } \Omega, \tag{4.4.54}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{4.4.55}$$

$$\mathbf{u}(0,y) = \mathbf{u}_*(y) \quad \text{for } y \in (0, L), \tag{4.4.56}$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0} \quad \text{for } x \in (0,\infty), \tag{4.4.57}$$

$$|\mathbf{u}(x,y)| \to 0 \quad \text{as } x \to \infty \text{ and } \ y \in (0,L), \tag{4.4.58}$$

where the forces field satisfies (4.2.8)-(4.2.10). In Section 4.3 (see (4.3.40)) has been established the existence of a weak solution $\mathbf{u}$ having a finite global energy

$$E := \int_\Omega \left(|\nabla\mathbf{u}|^2 + |u|^{1+\sigma(\theta)}\right)\, d\mathbf{x} \tag{4.4.59}$$

and consequently, from (4.2.9) and assuming[10] that $|u| \leq 1$,

$$\mathcal{E} := \int_\Omega \left(|\nabla\mathbf{u}|^2 + |u|^{1+\sigma^+}\right)\, d\mathbf{x} < \infty. \tag{4.4.60}$$

[10] Otherwise the stopping effect does not occur.

Theorem 4.4.1 *Assume that* $\mathbf{f}$ *satisfies (4.2.8)-(4.2.10) and* $\mathbf{u}$ *is any weak solution of the problem (4.4.54)-(4.4.58), with a finite global energy (4.4.59) and where* θ *is given by (4.4.53). Then* $\mathbf{u}(x,y) = \mathbf{0}$ *for* $x > a'$, *where* $a' = a'(E, L, \delta, \nu, \sigma^+)$ *is a positive constant.*

The proof of this theorem follows exactly the same reasoning of the proof of Theorem 3.3.1.

As in Sections 2.5 and 3.3, we introduce the associated stream function ψ (see (2.3.15)) and we reduce the study of problem (4.4.54)-(4.4.58), to the consideration of the following fourth-order problem where the pressure term does not appear anymore,

$$\nu\triangle^2\psi + \tfrac{\partial f_1}{\partial y} - \tfrac{\partial f_2}{\partial x} = \psi_y\triangle\psi_x - \psi_x\triangle\psi_y \quad \text{in } \Omega, \tag{4.4.61}$$

$$\psi(x,0) = \psi(x,L) = \tfrac{\partial\psi}{\partial n}(x,0) = \tfrac{\partial\psi}{\partial n}(x,L) = 0 \quad \text{for } x \in (0,\infty), \tag{4.4.62}$$

$$\psi(0,y) = \textstyle\int_0^y u_*(s)ds, \;\; \tfrac{\partial\psi}{\partial n}(0,y) = v_*(y) \quad \text{for } y \in (0,L), \tag{4.4.63}$$

$$\psi(x,y),\; |\nabla\psi(x,y)| \to 0, \text{ as } x \to \infty \quad \text{and for } y \in (0,L). \tag{4.4.64}$$

Here $\mathbf{f} = (f_1, f_2) = (f_1(\mathbf{x}, \theta, \psi_y, -\psi_x), f_2(\mathbf{x}, \theta, \psi_y, -\psi_x))$ and we recall that θ is assumed to be given (see (4.4.53)). The notion of weak solution is adapted again to the information we have on the function $\mathbf{f}$.

Definition 4.4.1 *Given* θ *satisfying (4.4.53), a function* ψ *is a weak solution of problem (4.4.61)-(4.4.64), if:*
(i) $\psi \in \mathrm{H}^2(\Omega)$, $\mathbf{f}(\mathbf{x}, \theta, \psi_y, -\psi_x) \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$;
(ii) $\psi(0,y) = \int_0^y u_*(s)ds$, $\frac{\partial\psi}{\partial n}(0,y) = v_*(y)$, $\psi(x,0) = \psi(x,L) = \frac{\partial\psi}{\partial n}(x,0) = \frac{\partial\psi}{\partial n}(x,L) = \psi(0,L) = 0$, *and* ψ, $|\nabla\psi| \to 0$, *when* $x \to \infty$;
(iii) For every $\phi \in \mathrm{H}^2_0(\Omega) \cap \mathrm{W}^{1,\infty}(\Omega)$ *with compact support,*

$$\nu\int_\Omega \triangle\psi\triangle\phi \, d\mathbf{x} - \int_\Omega (f_1\phi_y - f_2\phi_x)\, d\mathbf{x} = \int_\Omega \triangle\psi\,(\psi_x\phi_y - \psi_y\phi_x)\, d\mathbf{x}. \tag{4.4.65}$$

To establish the localization effect, we proceed as in Sections 2.5 and 3.3 and we prove the followings lemmas.

Lemma 4.4.1 *Given* θ *satisfying (4.4.53), if* $\mathbf{u}$ *is a weak solution of (4.2.11)-(4.2.18) in the sense of (ii) of Definition 4.3.1, then* ψ, *given by (2.3.15), is a weak solution of (4.4.61)-(4.4.64) in the sense of Definition 4.4.1.*

Lemma 4.4.2 *Given* θ *satisfying (4.4.53), let* ψ *be a weak solution of (4.4.61)-(4.4.64) with global energy (4.4.59) finite and assume that* $\mathbf{f}$ *satisfies (4.2.8)-(4.2.10). Then, for every* $a \geq x_g$ *and every positive integer* $m \geq 2$

$$\int_\Omega \left(\nu|\mathrm{D}^2\psi|^2 + \delta|\psi_y|^{1+\sigma^+}\right)(x-a)_+^m d\mathbf{x} \leq$$

$$2m\nu\int_\Omega |\triangle\psi|\,|\psi_x|\,(x-a)_+^{m-1}d\mathbf{x} + 2m\nu\int_\Omega |\psi_y|\,|\psi_{xy}|\,(x-a)_+^{m-1}d\mathbf{x} \tag{4.4.66}$$

$$+m(m-1)\nu\int_\Omega |\triangle\psi|\,|\psi|\,(x-a)_+^{m-2}d\mathbf{x} + m\int_\Omega |\triangle\psi|\,|\psi_y|\,|\psi|\,(x-a)_+^{m-1}d\mathbf{x}.$$

From the left-hand side of (4.4.66), it will arise the energy type term which depends on a

$$\mathcal{E}_m(a) = \int_\Omega \left(|\mathrm{D}^2\psi|^2 + |\psi_y|^{1+\sigma^+} \right) (x-a)_+^m d\mathbf{x}$$

and we again observe that

$$\mathcal{E}_0(0) = \mathcal{E}, \quad (\mathcal{E}_m(a))^{(k)} = (-1)^k \frac{m!}{(m-k)!} \mathcal{E}_{m-k}(a), \ 0 \leq k \leq m.$$

Then, the following lemma is proved as in Section 3.3.

Lemma 4.4.3 *Let ψ be a weak solution of (4.4.61)-(4.4.64) and let us assume* **f** *satisfies (4.2.8)-(4.2.10). Then, the following differential inequality holds for $a \geq x_g$ (x_g is given in (4.2.10)):*

$$\mathcal{E}_m(a) \leq C\left(\mathcal{E}_{m-p}(a)\right)^{\mu_1} + C\left(\mathcal{E}_{m-p}(a)\right)^{\mu_2}$$

for every integer $m > 3$, where $C = C(L, m, \delta, \nu, \sigma^+)$ are different positive constants and $\mu_j = \mu_j(m, \sigma^+) > 1$, $j = 1, 2$. Moreover, $\mathcal{E}_2(a) < \infty$ for any $a \geq x_g$. In fact,

$$\mathcal{E}_2(a) \leq C\mathcal{E}_0(a) + C\left(\mathcal{E}_0(a)\right)^{\mu_1} + C\left(\mathcal{E}_0(a)\right)^{\mu_2},$$

where C are different positive constants, the first an universal constant and the others such that $C = C(L, \delta, \nu, \sigma^+)$, and $\mu_j = \mu_j(\sigma^+) > 1$, $j = 1, 2$.

Proof: [Theorem 4.4.1] We take $m = 4$ in Lemma 4.4.3 and then we have the fractional differential inequality

$$\mathcal{E}_4(a) \leq C\left(\mathcal{E}_2(a)\right)^{\mu_1} + C\left(\mathcal{E}_2(a)\right)^{\mu_2},$$

where, according to what we have done in Section 3.3, $\mu_j = \mu_j(\sigma^+) > 1$, $j = 1, 2$ and $C = C(L, m, \delta, \nu, \sigma^+)$ means two different positive constants. Using Lemma 4.4.3 with $m = 2$ and because of the finiteness of $\mathcal{E}$ (see (4.4.60)), we can easily see that $\mathcal{E}_2(a)$ is finite. Then, using Lemma 3.3.4 and proceeding as in Section 3.3, we prove the support of $\mathcal{E}_0(a)$ is a bounded interval $[0, a^*]$ with $a^* \leq a'$, where a' is an upper limit to a^* and given by

$$a' = \frac{15 + \sigma^+}{1 - \sigma^+} C^{\frac{4}{7+\sigma^+}} \mathcal{E}^{\frac{1}{2(7+\sigma^+)}}, \quad C = C(E, L, \delta, \nu, \sigma^+).$$

Then $\mathcal{E}_0(a) = 0$ for $a > a'$, which implies $\mathbf{u} = \mathbf{0}$ almost everywhere for $x > a'$.□

Remark 4.4.1 *Remarks 2.5.2, 2.5.3, 3.3.1 and 3.3.2 are valid for the stationary Boussinesq problem.*

Remark 4.4.2 *We can carry out, for this case, the result establishing the stagnation effect proved in Section 2.6. But, in this case, we assume (4.2.8)-(4.2.9) and (4.2.10) is replaced by*

$$|f_2(\mathbf{x}, \theta, \mathbf{u})| \leq \gamma(x_s - x)_+^\zeta \tag{4.4.67}$$

for some $x_s > 0$ and some positive constants γ and ζ. Moreover, these localizations effects are valid if we consider localized forces field as in Section 2.7 (cf. Antontsev et al [10]).

4.5 Case of a Temperature Depending Viscosity

A harder, but very interesting, problem arises when the viscosity depends also on the temperature, which is very often the case in many concrete applications. In this case, the equation of motion (4.4.54) must be replaced by

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \text{div}\ (2\nu(\theta)\mathbf{D}) - \nabla p + \mathbf{f}(\mathbf{x},\theta,\mathbf{u}), \tag{4.5.68}$$

where $\mathbf{D}$ is the rate of strain tensor (1.4.13). We assume that

$$0 < \nu^- \leq \nu(\theta) \leq \nu^+ < \infty, \tag{4.5.69}$$

for some constants ν^- and ν^+, and the equation (4.3.22) of (ii) of Definition 4.3.1 is replaced by

$$2\int_\Omega \nu(\theta)\mathbf{D} : \nabla\varphi\, d\mathbf{x} + \int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\varphi\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\varphi\, d\mathbf{x}. \tag{4.5.70}$$

We assume the existence of, at least, one weak solution $(\theta,\mathbf{u})$, in the sense of Definition 4.3.1 with (4.3.22) replaced by (4.5.70), to problem {(4.5.68), (4.2.12)-(4.2.18)} having a finite global energy (4.4.59). To establish the localization effect, we proceed as in the proof of the above theorem by introducing the stream function (2.3.15) associated with the vector velocity and we reduce the problem {(4.5.68),(4.4.55)-(4.4.58)} to the following one

$$[\nu(\theta)\,(\psi_{xx}-\psi_{yy})]_{xx} + [\nu(\theta)\,(\psi_{yy}-\psi_{xx})]_{yy} + 4\,[\nu(\theta)\psi_{xy}]_{xy} \tag{4.5.71}$$

$$+\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = \psi_y\triangle\psi_x - \psi_x\triangle\psi_y$$

$$\psi(x,0) = \psi(x,L) = \tfrac{\partial\psi}{\partial n}(x,0) = \tfrac{\partial\psi}{\partial n}(x,L) = 0 \quad \text{for } x\in(0,\infty), \tag{4.5.72}$$

$$\psi(0,y) = \textstyle\int_0^y u_*(s)ds,\ \tfrac{\partial\psi}{\partial n}(0,y) = v_*(y) \quad \text{for } y\in(0,L), \tag{4.5.73}$$

$$\psi(x,y),\ |\nabla\psi(x,y)| \to 0,\ \text{as } x\to\infty \quad \text{and for } y\in(0,L), \tag{4.5.74}$$

where again $\mathbf{f} = (f_1,f_2) = (f_1(\mathbf{x},\theta,\psi_y,-\psi_x), f_2(\mathbf{x},\theta,\psi_y,-\psi_x))$. The notion of weak solution to problem (4.5.71)-(4.5.74) is adapted, from Definition 4.4.1, by replacing (4.4.65) by

$$\int_\Omega \nu(\theta)\left[(\psi_{xx}-\psi_{yy})\,(\phi_{xx}-\phi_{yy}) + 4\psi_{xy}\phi_{xy}\right] d\mathbf{x}$$

$$-\int_\Omega (f_1\phi_y - f_2\phi_x)\ d\mathbf{x} = \int_\Omega \triangle\psi\,(\psi_x\phi_y - \psi_y\phi_x)\ d\mathbf{x}.$$

In this case, the counterpart of (4.4.66) is

$$\int_\Omega \left(\nu^-|\mathrm{D}^2\psi|^2 + \delta|\psi_y|^{1+\sigma^+}\right)(x-a)_+^m d\mathbf{x} \leq$$

$$2m\nu^+\int_\Omega (|\psi_{xx}|+|\psi_{yy}|)|\psi_x|(x-a)_+^{m-1}d\mathbf{x} + 2m\nu^+\int_\Omega |\psi_{xy}||\psi_y|(x-a)_+^{m-1}d\mathbf{x}$$

$$+m(m-1)\nu^+\int_\Omega(|\psi_{xx}|+|\psi_{yy}|)|\psi|(x-a)_+^{m-2}d\mathbf{x}$$

$$+m\int_\Omega|\triangle\psi||\psi_y||\psi|(x-a)_+^{m-1}d\mathbf{x}.$$

Proceeding as in the proof of the above theorem and using the assumptions (4.2.9) and (4.5.69), we obtain the same localization effect.

4.6 Exponential Decay for the Temperature

In the previous sections we have shown that, wether the viscosity depends on the temperature or not, we obtain $\mathbf{u}=\mathbf{0}$ almost everywhere for $x>a'$, where a' is given in Theorem 4.4.1. Thus, in both cases, the Boussinesq system reduces to

$$\triangle\theta=0 \quad \text{in } \Omega_{a'}=(a',\infty)\times(0,L),$$

$$\theta=\theta_* \quad \text{on } y=0,\ L,$$

$$\theta(x,y)\to 0 \quad \text{as } x\to\infty \text{ and } \ y\in(0,L),$$

with an unknown temperature θ on $x=a'$. From Definition 4.3.1-(i), we see that a weak solution θ should has to satisfy

$$\int_{\Omega_{a'}}\nabla\theta\cdot\nabla\zeta\,d\mathbf{x}=0. \tag{4.6.75}$$

for every test function $\zeta\in\mathrm{H}_0^1(\Omega)$. Then, we take in (4.6.75), as a test function, $\zeta=\theta\,\rho_k(x)$, where θ is a weak solution for this problem and $\rho_k(x)$ is the approximating sequence of $(x-a)_+^m$, with $m\geq 2$ an integer, defined on page 35. Here, the parameter a is taken such that $a>a'$ and a' is given in Theorem 4.4.1. Proceeding in this way and then applying the Lebesgue convergence theorem, we obtain

$$\int_{\Omega_{a'}}|\nabla\theta|^2(x-a)_+^m\,d\mathbf{x}=-m\int_{\Omega_{a'}}\theta\,\theta_x\,(x-a)_+^{m-1}\,d\mathbf{x}.$$

Applying modules, Cauchy's inequality with a suitable ε and Poincaré's inequality, we arrive at

$$\int_{\Omega_{a'}}\theta^2(x-a)_+^m\,d\mathbf{x}\leq C\int_{\Omega_{a'}}\theta^2(x-a)_+^{m-2}\,d\mathbf{x}, \quad C=C(L,m).$$

Then, we can only derive an exponential decay for the temperature. For instance, taking $m=2$, we obtain

$$\int_{\Omega_{a'}}\theta^2\,d\mathbf{x}\leq C\exp(-Ca) \quad \text{for every } a>a'.$$

where C means two different positive constants (see also Remark 2.5.4 on page 41).

Chapter 5

Evolutionary Navier-Stokes Problem

"If in the course of time, the external forces die out, and if the boundary conditions correspond to a state of rest, then the motion also dies out, regardless of what the motion was at the initial instant of time."

O.A. Ladyzhenskaya [72, p. 4].

This chapter deals with the time dependent Navier-Stokes problem which is introduced in Section 5.1. The results presented in this chapter do not answer completely to all the questions we would like. In fact we are still working on this problem. Nevertheless, many results on existence and uniqueness of a weak solution, as well the localization effects in time, wether the domain is bounded or unbounded, are already possible to show. We present these results either in their complete rigorous form, the case of localization effects in time, in Section 5.3, or just as simple statements, the case of existence and uniqueness of a weak solution, in Section 5.2, which proofs, if are not given, are addressed to the article in preparation by Antontsev et al [11]. Moreover, in Section 5.4, some localization effects in time for the Cauchy problem are established.

5.1 Introduction

We consider the *evolutionary Navier-Stokes system*

$$\mathbf{u}_t - \nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p, \tag{5.1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \tag{5.1.2}$$

in the domain $Q = \Omega \times (0,T)$, $0 < T \leq \infty$, where $\Omega = (0,\infty)\times(0,L)$, $L > 0$, and ν is the *kinematic viscosity coefficient*. Navier-Stokes system (5.1.1)-(5.1.2) is derived from Navier-Stokes system {(1.4.22), (1.4.11), (1.4.23)} by assuming the fluid is homogeneous, the kinematic viscosity is constant and by considering a flow with only two velocity components: $u(x,y,t)$ and $v(x,y,t)$. To the Navier-Stokes system (5.1.1)-(5.1.2), we append a possible non-zero initial velocity

$$\mathbf{u} = \mathbf{u}_0, \quad \mathbf{x}\in\Omega,\ t = 0. \tag{5.1.3}$$

and we prescribe zero velocities at the strip entrance and on the lateral walls

$$\mathbf{u}(0,y,t) = \mathbf{0},\ y\in(0,L),\ t\in(0,T), \tag{5.1.4}$$

$$\mathbf{u}(x,0,t) = \mathbf{u}(x,L,t) = \mathbf{0},\ x\in(0,\infty),\ t\in(0,T), \tag{5.1.5}$$

as well at infinity

$$\mathbf{u}(x,y,t)\to\mathbf{0},\ \text{as } x\to\infty \text{ and } \ y\in(0,L),\ t\in(0,T). \tag{5.1.6}$$

If are considering $\mathbf{u} = \mathbf{u}_*$ on $x = 0$, we add the compatibility conditions to this system

$$\int_0^L u_*(s,t)\,ds = 0, \quad \text{for every } t\in(0,T)$$

and

$$\mathbf{u}_*(0,t) = \mathbf{u}_*(L,t) = \mathbf{0}, \quad \text{for every } t\in(0,T).$$

Here, because we are dealing with an evolutionary problem, we have two main different localization effects to study. The first, is the localization effect in space (cf. Definition 1.2.3) and is related with the study made in the previous chapters, now for the evolutionary Navier-Stokes problem. Thus, we assume the body forces field is given in a nonlinear feedback form, $\mathbf{f} : Q\times\mathbb{R}^2\to\mathbb{R}^2$, $\mathbf{f} = \mathbf{f}(\mathbf{x},t,\mathbf{u}) = (f_1(\mathbf{x},t,\mathbf{u}), f_2(\mathbf{x},t,\mathbf{u}))$, such that, for every $\mathbf{u}\in\mathbb{R}^2$, $\mathbf{u} = (u,v)$, and for almost all $(\mathbf{x},t)\in Q$,

$$-\mathbf{f}(\mathbf{x},t,\mathbf{u})\cdot\mathbf{u} \geq \delta\,|u|^{1+\sigma} - g(\mathbf{x},t) \tag{5.1.7}$$

for some $\delta > 0$, $\sigma\in(0,1)$ and for some function

$$g\in \mathrm{L}^2\left(0,T;\mathrm{L}^1(\Omega^{x_g})\right),\ g\geq 0,\ g(\mathbf{x},t) = 0 \ \text{ a.e. in } \ Q_{x_g}, \tag{5.1.8}$$

where $Q_{x_g} = (0,T)\times\Omega_{x_g}$, $\Omega^{x_g} = (0,x_g)\times(0,L)$ and $\Omega_{x_g} = (x_g,\infty)\times(0,L)$, with $0\leq x_g<\infty$.

We notice that even for the Navier-Stokes system with zero external forces field, we have not found any direct reference where is proved an exponential decay analogous to (2.2.8). We only have found two works, one of Ames et al [2] and the other of Lin [78], but for the Stokes problem, establishing an exponential decay in terms of the distance from the finite end of the pipe or the strip for the three or two dimensional problem, respectively. There, were used the same tools of Horgan [62], previously used by Knowles [68] and Toupin [117] in their energy approach to the Saint-Venant's Principle[1]. It should be noted that Elcrat and Sigillito [41] actually looked at the question of spatial decay for the evolutionary Navier-Stokes equations, but their method required an assumption on an auxiliary function that is not generally satisfied. In our opinion the reason for the existence of few works in this direction, is because the study of spatial decay of solutions of time dependent problems is of relatively recent origin.

The second study we want to carry out, is the localization effect in time (cf. Definition 1.2.2) and intends to answer if it is possible or not to stop, in a finite time, a fluid driven by (5.1.1)-(5.1.6), with the body forces field satisfying (5.1.7)-(5.1.8). We notice that for the linear Navier-Stokes system, the best we can get is an exponential time decay

$$\|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq C_1 \exp(-C_2 t) \quad \text{for all } t \in (0,T), \tag{5.1.9}$$

where $C_1 = C_1(\mathbf{f}, \mathbf{u}_0, \nu)$ and $C_2 = C_2(\nu, L)$ (cf. Sohr [110, §3.4-5]). See also Ladyzhenskaya [72, §6.5]) for the behavior of solutions as $t \to \infty$ and Kozono and Ogawa [69], and Borchers and Myakawa [23] for $\mathbf{L}^2$ decays in unbounded domains.

It should be remarked that questions of decay, in time and in space, of solutions to evolutionary Navier-Stokes equations in different norms have been studied, among others, by the aforementioned authors, Knightly [67], Schonbeck [101, 102], Takahashi [113] and Wiegner [122]. For instance, in a article by Amrouch et al [4], which extend the results by Schonbeck [101, 102] and Takahashi [113], is proved that the (strong) solutions of the Navier-Stokes equations, with zero external forces field, which decay at the (time) rate $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C(t+1)^{-\mu}$, will have the following pointwise space-time decay

$$|\mathrm{D}^\alpha \mathbf{u}| \leq C \frac{1}{(t+1)^{\rho_0}(1+|\mathbf{x}|^2)^{k/2}},$$

where $\rho_0 = (1-2k/N)\,(m/2+\mu+N/4)$, $|\alpha| = m$, $\mu > N/4$ and $C = C(k,m)$. Takahashi [113] obtained similar results by considering zero initial data and a non-zero external forces field satisfying $|\mathbf{f}(\mathbf{x},t)| \leq 1/(|\mathbf{x}|^\mu t^{-\lambda})$, as $|\mathbf{x}|$ and t tend to infinity, for suitable positive constants μ and λ.

5.2 Weak Formulation

In what concerns to the weak formulation of this problem, we search for solutions such that, additionally to (5.1.6), satisfy

$$\int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \, d\mathbf{x} \, ds < \infty \quad \text{for all } t \in (0,T).$$

[1]See Remarks 2.5.4 and 3.3.1.

Due to the fact that Poincaré's inequality (1.1.3) holds, our searched solution will be an element of the Bochner space $L^2(0,T;\mathbf{H}^1(\Omega))$. For the definition of weak solution, we need to recover the solenoidal vector spaces $\mathbf{H}(\Omega)$ and $\widetilde{\mathbf{H}}_0(\Omega)$ defined in Section 2.4.

Definition 5.2.1 *We say that* $\mathbf{u}$ *is a weak solution of problem (5.1.1)-(5.1.6), if:*
(i) $\mathbf{u} \in L^2(0,T;\widetilde{\mathbf{H}}_0(\Omega)) \cap L^\infty(0,T;\widetilde{\mathbf{H}}_0(\Omega))$, $\mathbf{u}_t \in L^2(0,T;\widetilde{\mathbf{H}}^{-1}(\Omega))$ *and* $\mathbf{f}(\mathbf{x},t,\mathbf{u}) \in L^2(0,T;\mathbf{L}^1_{\text{loc}}(\Omega))$*;*
(ii) $\|\mathbf{u}(\mathbf{x},t) - \mathbf{u}_0(\mathbf{x})\|_{\mathbf{L}^2(\Omega)} \to 0$*, when* $t \to 0$*;*
(iii) For every $\varphi \in C^1(0,T;\widetilde{\mathbf{H}}_0(\Omega)) \cap L^2(0,T;\mathbf{L}^\infty(\Omega))$ *with compact support*[2] *in* Q_T

$$\int_\Omega \mathbf{u}(\cdot,t)\cdot\varphi\,d\mathbf{x} - \int_0^t\int_\Omega \mathbf{u}\cdot\varphi_s\,d\mathbf{x}\,ds + \nu\int_0^t\int_\Omega \nabla\mathbf{u}:\nabla\varphi\,d\mathbf{x}\,ds$$
$$+\int_0^t\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\varphi\,d\mathbf{x}\,ds = \int_0^t\int_\Omega \mathbf{f}\cdot\varphi\,d\mathbf{x}\,ds + \int_\Omega \mathbf{u}_0\cdot\varphi_0\,d\mathbf{x},$$

for all $t \in [0,T)$*.*

For the lack of time and space, we will assume, without proof, the existence of a weak solution to problem (5.1.1)-(5.1.6) in the sense of Definition 5.2.1, where the forces field is given by $\mathbf{f}: Q\times\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f}(\mathbf{x},t,\mathbf{u}) = (f_1(\mathbf{x},t,\mathbf{u}), f_2(\mathbf{x},t,\mathbf{u}))$, with

$$\mathbf{f}(\mathbf{x},t,\mathbf{u}) = -\delta\left(|u|^{\sigma-1}u,0\right) - \mathbf{h}(\mathbf{x},t,\mathbf{u})$$

for some $\delta > 0$, $0 < \sigma < 1$ and some Carathéodory function[3] $\mathbf{h}(\mathbf{x},\mathbf{t},\mathbf{u})$ such that

$$\mathbf{h}(\mathbf{x},\mathbf{t},\mathbf{u})\cdot\mathbf{u} \geq -g(\mathbf{x},t)$$

for every $\mathbf{u} \in \mathbb{R}^2$ and almost all $(\mathbf{x},t) \in Q$ and for some function g satisfying (5.1.8). The result on existence is carried out by assuming $\mathbf{u}_0 \in \mathbf{H}(\Omega)$ and proceeding as we did in Section 3.2 but in the Bochner spaces framework (see *e.g.* Galdi [51, §3] and Temam [115, §III.3]). Another idea, consists in using a semi-discretization in time (cf. Feistauer [45, §8.7] and Temam [115, §III.4]) converting the evolutionary problem into a sequence of stationary problems for which we can use the results of Section 3.2. This procedure allows us to construct a sequence of approximate solutions from which we can extract a subsequence weakly convergent to a weak solution of the original problem. Moreover, if the non-increasing condition

$$(\mathbf{f}(\mathbf{x},t,\mathbf{u}_1) - \mathbf{f}(\mathbf{x},t,\mathbf{u}_2))\cdot(\mathbf{u}_1-\mathbf{u}_2) \leq 0 \tag{5.2.10}$$

holds for every $\mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^2$ and almost all $(\mathbf{x},t) \in Q$, we can prove that problem (5.1.1)-(5.1.6) has exactly one weak solution, by using the known uniqueness result for the evolutive two dimensional problem (see *e.g.* Galdi [51, §4] and Temam [115, §III.3]). If we assume the body forces field is given by the simpler expression

$$\mathbf{f}(\mathbf{x},t,\mathbf{u}) = -\delta(|u|^{\sigma-1}u,0), \tag{5.2.11}$$

[2] Notice that $\varphi(\cdot,0)$ need not be zero (cf. Galdi [51, §2])

[3] $\mathbf{h}(\mathbf{x},t,\mathbf{u})$ is measurable in $(\mathbf{x},t)$ for every $\mathbf{u} \in \mathbb{R}^2$ and continuous in $\mathbf{u}$ for almost all $(\mathbf{x},t) \in Q$.

we can drop the non-increasing condition (5.2.10) to prove the uniqueness result. For the detailed proofs of these results, we address the reader to the article in preparation by Antontsev et al [11].

To establish the localization effects, the procedure is the same as we did in the previous chapters. We introduce the associated stream function $\psi(\mathbf{x},t)$ defined by

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in} \quad Q. \tag{5.2.12}$$

This function is defined up to an arbitrary function of the variable time, which we fix it by considering $\psi(\mathbf{0},t) = 0$. By classical methods (see *e.g.* Ladyzhensakaya [72, §6.7] or Marion and Temam [84, §6]), we reduce the problem (5.1.1)-(5.1.6) to the consideration of the problem posed by the following equations

$$-\triangle\psi_t + \nu\triangle^2\psi + \tfrac{\partial f_1}{\partial y} - \tfrac{\partial f_2}{\partial x} = \psi_y\triangle\psi_x - \psi_x\triangle\psi_y \quad \text{in } Q_T, \tag{5.2.13}$$

$$\psi(\mathbf{x},0) = \textstyle\int_0^y u_0(x,s)ds \quad \text{for } \mathbf{x}\in\Omega, \tag{5.2.14}$$

$$\psi(x,0,t) = \psi(x,L,t) = 0 \quad \text{for } x\in(0,\infty)\,,\ t\in(0,T), \tag{5.2.15}$$

$$\tfrac{\partial}{\partial n}\psi(x,0,t) = \tfrac{\partial}{\partial n}\psi(x,L,t) = 0 \quad \text{for } x\in(0,\infty)\,,\ t\in(0,T), \tag{5.2.16}$$

$$\psi(0,y,t) = 0 \quad \text{for } y\in(0,L)\,,\ t\in(0,T), \tag{5.2.17}$$

$$\tfrac{\partial}{\partial n}\psi(0,y,t) = 0 \quad \text{for } y\in(0,L)\,,\ t\in(0,T), \tag{5.2.18}$$

$$\psi(\mathbf{x},t),\ |\nabla\psi(\mathbf{x},t)| \to 0, \quad \text{as } x\to\infty \text{ and } y\in(0,L)\,,\ t\in(0,T), \tag{5.2.19}$$

where the pressure term does not appear anymore and $\mathbf{f} = (f_1,f_2) = (f_1(\mathbf{x},t,\psi_y, -\psi_x), f_2(\mathbf{x},t,\psi_y,-\psi_x))$. Once again, the notion of weak solution is adapted to the information we have on the function $\mathbf{f}(\mathbf{x},t,\psi_y,-\psi_x)$.

Definition 5.2.2 *A function ψ is a weak solution of problem (5.2.13)-(5.2.19), if:*
(i) $\psi\in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$, $\psi_t\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, $\mathbf{f}(\mathbf{x},t,\psi_y,-\psi_x)\in\mathrm{L}^2(0,T;\mathbf{L}^1_{\mathrm{loc}}(\Omega))$;
(ii) $\psi(\mathbf{x},0) = \int_0^y u_0(x,s)ds$, $\psi(x,0,t) = \psi(x,L,t) = 0$, $\frac{\partial}{\partial n}\psi(x,0,t) = \frac{\partial}{\partial n}\psi(x,L,t) = 0$, $\psi(0,y,t) = 0$, $\frac{\partial}{\partial n}\psi(0,y,t) = 0$ and $\psi(\mathbf{x},t)$, $|\nabla\psi(\mathbf{x},t)|\to 0$ when $x\to\infty$;
(iii) For every $\phi\in\mathrm{C}(0,T;\mathrm{H}^2_0(\Omega))\cap\mathrm{L}^2(0,T;\mathrm{W}^{1,\infty}(\Omega))$ with compact support[4] in Q_T and such that $\phi_t\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$,

$$-\int_\Omega \triangle\psi(\cdot,t)\,\phi\,d\mathbf{x} + \int_0^t\int_\Omega \triangle\psi\,\phi_s\,d\mathbf{x}\,ds + \nu\int_0^t\int_\Omega\triangle\psi\,\triangle\phi\,d\mathbf{x}\,ds \tag{5.2.20}$$

$$-\int_0^t\int_\Omega (f_1\phi_y - f_2\phi_x)\,d\mathbf{x}\,ds = \int_0^t\int_\Omega \triangle\psi\,(\psi_x\phi_y - \psi_y\phi_x)\,d\mathbf{x}\,ds - \int_\Omega\triangle\psi_0\,\phi_0\,d\mathbf{x}$$

for all $t\in[0,T)$.

It can be proved (cf. Antontsev et al [11]) that if $\mathbf{u}$ is any weak solution of (5.1.1)-(5.1.6) in the sense of Definition 5.2.1, then ψ, given by (5.2.12), is a weak solution of (5.2.13)-(5.2.19) in the sense of Definition 5.2.2.

[4]Notice again that $\phi(\cdot,0)$ need not be zero.

5.3 Localization Effects

As for the localization effects and so far, the question of extending the localization effects (in space) proved in Sections 2.5, 2.6 and 3.3 to the Navier-Stokes problem (5.1.1)-(5.1.6), with the body forces field satisfying (5.1.7)-(5.1.8), remains open. At first sight, one can be lead to think the localization effect in space for this problem could be derived from that ones in Section 3.3 by reducing the evolutionary problem (5.1.1)-(5.1.6) to a stationary one by using a semi-discretization in time. Even in that case, either we discretize the Navier-Stokes problem (cf. Marion and Temam [84, §15]) or the correspondent reduced problem for the stream function (see *e.g.* Quarteroni and Valli [96]), we have found some difficulties in obtaining the desired localization effects. However, if it will be possible to obtain such localization effect, we know from the study of these localization effects in other time dependent problems (see Antontsev et al [12] and Díaz [34]), that the following assumption must be satisfied

$$\operatorname{supp}\mathbf{u}_0\cap\Omega_{x_g}=\emptyset,\quad \Omega_{x_g}=(0,x_g)\times(0,L),\tag{5.3.21}$$

where x_g is given in (5.1.8).

On the other hand, we can already prove a localization effect in time. Let us define the energy

$$E(t)=\frac{1}{2}\int_\Omega|\mathbf{u}|^2d\mathbf{x}=\frac{1}{2}\int_\Omega|\nabla\psi|^2d\mathbf{x}\tag{5.3.22}$$

and notice that, if $\mathbf{u}$ is any weak solution of (5.1.1)-(5.1.6), then (5.3.22) is finite for almost all $t\in(0,T)$. We notice also that for the time exponential decay, the assumption (5.3.21) is not needed.

Theorem 5.3.1 *Let* $\mathbf{u}$ *be a weak solution of problem (5.1.1)-(5.1.6) and let* $\mathbf{f}$ *satisfies (5.1.7)-(5.1.8) with* $x_g=0$*. Then, there exists* $t^*\in(0,T)$ *such that*

$$\mathbf{u}=\mathbf{0}\quad\textit{for all } t\in[t^*,T)\quad\textit{and}\quad\textit{for almost all } \mathbf{x}\in\Omega.\tag{5.3.23}$$

Proof: Let ψ be a weak solution of (5.2.13)-(5.2.19) with a finite energy (5.3.22). We take ψ as a test function in (5.2.20) and we obtain

$$\frac{1}{2}\int_\Omega|\nabla\psi|^2\,d\mathbf{x}+\nu\int_0^t\int_\Omega|\mathrm{D}^2\psi|^2\,d\mathbf{x}\,ds\tag{5.3.24}$$
$$-\int_0^t\int_\Omega(f_1\psi_y-f_2\psi_x)\,d\mathbf{x}\,ds=\frac{1}{2}\int_\Omega|\nabla\psi_0|^2\,d\mathbf{x}$$

for all $t\in[0,T)$. Differentiating (5.3.24) with respect to t, then using the assumptions (5.1.7)-(5.1.8) with $x_g=0$, we obtain

$$\frac{d}{dt}E(t)+\int_\Omega\left(\nu|\mathrm{D}^2\psi|^2+\delta|\psi_y|^{1+\sigma}\right)d\mathbf{x}\le 0\quad\text{for all } t\in(0,T),\tag{5.3.25}$$

where $E(t)$ is given by (5.3.22). Next, we use Gagliardo-Nirenberg's inequality (1.1.5), with $N=2$, $j=1$, $k=2$, $p=2$, $q=2$ and $r=1+\sigma$, to obtain

$$2E(t)\le C\left\|\mathrm{D}^2\psi\right\|_{\mathrm{L}^2(\Omega)}^{2\theta}\|\psi\|_{\mathrm{L}^{1+\sigma}(\Omega)}^{2(1-\theta)},\quad C=C(\sigma),\quad \theta=\frac{2}{3+\sigma}.\tag{5.3.26}$$

Because $\psi \in \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))$, $\|\mathrm{D}^2\psi\|_{\mathbf{L}^2(\Omega)} < \infty$ and the right-hand side of (5.3.26) is finite if $\|\psi\|_{\mathrm{L}^{1+\sigma}(\Omega)} < \infty$ for all $t \in (0,T)$. But, this is true by the Sobolev imbedding $\mathrm{H}^2(\Omega) \to \mathrm{L}^{1+\sigma}(\Omega)$. In consequence, using Poincaré's inequality on the term $\|\psi\|_{\mathrm{L}^{1+\sigma}(\Omega)}$, and the algebraic inequality (1.1.1),

$$E(t) \le C\left(\int_\Omega \left(|\mathrm{D}^2\psi|^2 + |\psi_y|^{1+\sigma}\right) d\mathbf{x}\right)^\mu, \quad C = C(L,\sigma), \tag{5.3.27}$$

where, from Gagliardo-Nirenberg's inequality (5.3.26),

$$\mu = \theta + 2\frac{1-\theta}{1+\sigma} = 1 + \frac{1-\sigma}{3+\sigma} > 1 \ \ (0 < \sigma < 1). \tag{5.3.28}$$

From (5.3.25), it comes

$$\frac{d}{dt}E(t) + C\left(E(t)\right)^{\frac{1}{\mu}} \le 0, \quad C = C(L,\delta,\nu,\sigma) \tag{5.3.29}$$

whose explicit integration between 0 and $t > 0$ gives

$$E(t)^{\frac{\mu-1}{\mu}} \le E(0)^{\frac{\mu-1}{\mu}} - C\frac{\mu-1}{\mu}t. \tag{5.3.30}$$

Then $E(t) = 0$, and consequently (5.3.23) is satisfied, where from (5.3.28) and (5.3.30),

$$t^* = C^{-1}\frac{4}{1-\sigma}E(0)^{\frac{1-\sigma}{4}}, \tag{5.3.31}$$

which ends the proof.□

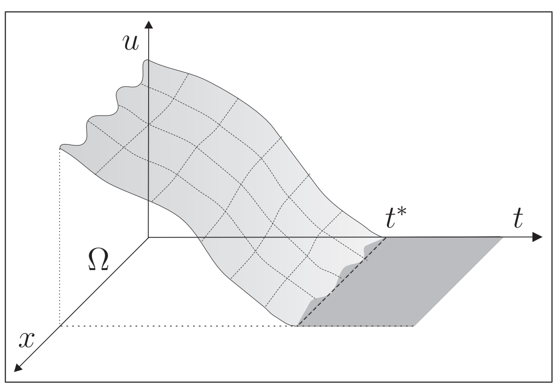

Figure 5.1: Stopping effect in time.

Remark 5.3.1 *Notice that the consideration of $x_g = 0$ in (5.1.8), corresponds to say that, in a certain sense, we are assuming the body forces field satisfies (5.2.11). For $x_g > 0$ in (5.1.8), we are not able to prove this localization effect.*

Remark 5.3.2 *We could also have considered non-homogeneous boundary conditions, say $\mathbf{u}_*$, on $x = 0$. But then, in order to carry out the same localization effect, we would have to assume the existence of a $t_* > 0$ such that $\mathbf{u}_* = \mathbf{0}$ for all $t \geq t_*$ and $E(t_*) < \infty$. In the above proof we only would have to replace the time $t = 0$ by $t = t_*$.*

Remark 5.3.3 *The above localization effect can be extended to the limit case of $\sigma = 0$. But, for $\sigma = 1$, we obtain from (5.3.29)*

$$\frac{d}{dt}E(t) + CE(t) \leq 0, \quad C = C(L, \delta, \nu, \sigma)$$

which implies the exponential decay $E(t) \leq C_1 \exp(-C_2 t)$, for all $t \in [0, T)$, with $C_1 = C_1(E(0))$ and $C_2 = C_2(L, \delta, \nu, \sigma)$. This is the same that was obtained for the Navier-Stokes system with prescribed linear forces field (see (5.1.9)).

Remark 5.3.4 *We also can extended this result for a general unbounded subdomain of $\mathbb{R}^2$ for which Poincaré's inequality is valid and, instead of (5.1.7)-(5.1.8), the forces field are assuming to satisfy the equivalent to (2.3.14),* i.e., *for every $\mathbf{u} \in \mathbb{R}^2$ and almost all $(\mathbf{x}, t) \in Q_T$*

$$-\mathbf{f}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{u} \geq \delta |\mathbf{u}|^{1+\sigma} \quad \textit{for some } \sigma \in (0, 1) \textit{ and } \delta > 0.$$

Is this case, the proof is analogous, only replacing $|\psi|^{1+\sigma}$ in (5.3.25) and (5.3.27) by $|\nabla\psi|^{1+\sigma}$ and justifying the finiteness of the right-hand side of (5.3.26) (with $|\nabla\psi|^{1+\sigma}$ instead of $|\psi|^{1+\sigma}$) with the Sobolev imbedding $\mathrm{H}^2(\Omega) \to \mathrm{W}^{1,1+\sigma}(\Omega)$. We notice that this result was derived in Antontsev et al [12, §4.7.4] for a non-Newtonian fluid[5]. But there, was not considered the case of two-dimensional Navier-Stokes equations.

Now, we consider the case when Ω is a finite strip $(0, R) \times (0, L)$, $0 < L$, $R < \infty$. In this case, instead of (5.1.6), we consider

$$\mathbf{u}(R, y, t) = \mathbf{0}, \; y \in (0, L), \; t \in (0, T), \tag{5.3.32}$$

which gives in the reduced problem

$$\psi(R, y, t) = \frac{\partial \psi}{\partial n}(R, y, t) = 0, \; y \in (0, L), \; t \in (0, T),$$

instead of (5.2.19).

Theorem 5.3.2 *Let $\mathbf{u}$ be a weak solution of problem {(5.1.1)-(5.1.5), (5.3.32)} and let $\mathbf{f}$ satisfies (5.1.7)-(5.1.8) with $x_g = 0$. Then, there exists $t^* \in (0, T)$ such that*

$$\mathbf{u} = \mathbf{0} \quad \textit{for all } t \in [t^*, T) \quad \textit{and} \quad \textit{for almost all } \mathbf{x} \in \Omega. \tag{5.3.33}$$

[5] See Section 2.3, page 23 (of this thesis).

Proof: Using Gagliardo-Nirenberg's inequality (1.1.6) instead of (1.1.5) and proceeding in the same way such as in the proof of Theorem 5.3.1, we obtain

$$\sqrt{2E(t)} \leq C_1 \left\| \mathrm{D}^2 \psi \right\|^{\theta}_{\mathrm{L}^2(\Omega)} \left\| \psi_y \right\|^{1-\theta}_{\mathrm{L}^{1+\sigma}(\Omega)} + C_2 \left\| \psi_y \right\|_{\mathrm{L}^{\tilde{q}}(\Omega)} \quad \text{for any } \tilde{q} \geq 1, \tag{5.3.34}$$

where θ is given in (5.3.26) and $C_i = C_i(L, \sigma, \widetilde{q})$, $i = 1, 2$, are positive constants. Because $\psi \in \mathrm{L}^2(0, T; \mathrm{H}^2(\Omega))$, the right-hand side of (5.3.34) is finite if $\|\psi\|_{\mathrm{L}^{1+\sigma}(\Omega)} < \infty$ and $\|\psi\|_{\mathrm{L}^{\tilde{q}}(\Omega)} < \infty$. But, by Hölder's inequality, this is true. Choosing $\tilde{q} = 2/[\theta(\sigma - 1) + 2] = (3+\sigma)/(2+\sigma)$ $(0 < \sigma < 1 \Rightarrow (1+\sigma)/\tilde{q} > 1)$, we obtain, after using Hölder's inequality on the second term of the right-hand side, (5.3.27), where now $C = C(\sigma, \Omega)$. Then following in the same way as in the above proof, we prove also that (5.3.33) holds, where t^* is given by (5.3.31), now with $C = C(\delta, \nu, \sigma, \Omega)$.□

Remark 5.3.5 *The localization effects presented in this section, are also valid if we consider time dependent localized forces field as we did in Section 2.7 (cf. Antontsev et al [11]).*

5.4 The Cauchy Problem

In this section, we consider the problem given by (5.1.1)-(5.1.3) in $Q = \Omega \times (0, T)$, where Ω is an unbounded subdomain of $\mathbb{R}^N$, $N \geq 3$, $0 < T \leq \infty$. Here we assume zero velocities on the compact boundary (if there exists)

$$\mathbf{u} = \mathbf{0} \quad \text{at } \partial\Omega \times (0, T). \tag{5.4.35}$$

as well at infinity

$$\mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \ \text{ and } \ t \in (0, T). \tag{5.4.36}$$

We assume[6] the body forces field $\mathbf{f} : Q \times \mathbb{R}^N \to \mathbb{R}^N$, satisfies for every $\mathbf{u} \in \mathbb{R}^N$ and for almost all $(\mathbf{x}, t) \in Q$,

$$-\mathbf{f}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{u} \geq \delta \left| \mathbf{u} \right|^{1+\sigma} \quad \text{for some } \delta > 0 \text{ and } 0 < \sigma < 1. \tag{5.4.37}$$

In this case, the notion of weak solution can be adapted from Definition 5.2.1, where now $\Omega \subseteq \mathbb{R}^N$, $\mathbf{H}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathrm{div}\mathbf{u} = 0\}$ and

$$\widetilde{\mathbf{H}}_0(\Omega) = \left\{ \mathbf{u} \in \mathbf{H}(\Omega) : \mathbf{u} = \mathbf{0} \text{ at } \partial\Omega, \ \lim_{|\mathbf{x}| \to \infty} \mathbf{u} = \mathbf{0} \right\}.$$

If $\Omega = \mathbb{R}^N$, we only assume (5.4.36) and the functional space $\widetilde{\mathbf{H}}_0(\Omega)$ is given without the conditions at the compact boundary, and that is known as the *Cauchy problem*. We assume the existence of, at least, one weak solution for this problem with the body forces field satisfying (5.4.37). We conjecture that this, as well the uniqueness, can be proved in the same way as for the two-dimensional aforementioned problem by using some known results with a prescribed linear forces field (cf. Galdi [51, §7]; see also Kato [66] and Ladyzhenskaya [72, §6.8]). In the case of linear forces field, we also can only obtain an exponential time decay as in (5.1.9) (cf. Ladyzhenskaya [72, §6.8]).

[6] This is exactly the same assumption as (2.3.14).

Theorem 5.4.1 *Let* $\mathbf{u}$ *be a weak solution of the problem given by (5.1.1)-(5.1.3) in* $Q = \Omega \times (0,T)$, *with* $\Omega \subset \mathbb{R}^N$ *unbounded* ($\Omega = \mathbb{R}^N$), $N \geq 3$, $0 < T \leq \infty$, *and by (5.4.35)-(5.4.36) (resp. (5.4.36)). Assume* $\mathbf{f}$ *satisfies (5.4.37),* $\mathbf{u} \in \mathbf{L}^{1+\sigma}(\mathbb{R}^N)$ *and* $E(0)$ *is finite. Then, there exists* $t^* \in (0,T)$ *such that*

$$\mathbf{u} = \mathbf{0} \quad \textit{for all } t \in [t^*, T) \quad \textit{and} \quad \textit{for almost all } \mathbf{x} \in \Omega. \tag{5.4.38}$$

Proof: We formally multiply (5.1.1) by $\mathbf{u}$, a weak solution of the problem given by (5.1.1)-(5.1.3), we integrate by parts over Ω using (5.1.2), (5.4.35)-(5.4.37) and the assumption (5.4.37), to obtain

$$\frac{d}{dt}E(t) + \int_\Omega \left(\nu|\nabla\mathbf{u}|^2 + \delta|\mathbf{u}|^{1+\sigma}\right) d\mathbf{x} \leq 0, \quad \text{for all } t \geq 0. \tag{5.4.39}$$

Then, we use a vectorial version of (1.1.5), with a general N, $j = 0$, $k = 1$, $p = 2$, $q = 2$ and $r = 1 + \sigma$, to obtain

$$2E(t) \leq C \left\|\nabla\mathbf{u}\right\|^{2\theta}_{\mathbf{L}^2(\Omega)} \left\|\mathbf{u}\right\|^{2(1-\theta)}_{\mathbf{L}^{1+\sigma}(\Omega)}, \quad \theta = \frac{N(1-\sigma)}{N(1-\sigma) + 2(1+\sigma)}, \tag{5.4.40}$$

where $C = C(N,\sigma)$. By hypothesis, $\mathbf{u} \in \mathrm{L}^2(0,T;\widetilde{\mathbf{H}}_0(\Omega))$ and then $\left\|\nabla\mathbf{u}\right\|_{\mathbf{L}^2(\Omega)} < \infty$ for all $t \in (0,T)$. But, it can be proved that the Sobolev imbedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^{1+\sigma}(\Omega)$, $0 < \sigma < 1$, is possible only if Ω is bounded in some direction (see *e.g.* Adams [1, §5.5]). Then the right-hand side of (5.4.40) is finite only if we, additionally, assume $\mathbf{u} \in \mathbf{L}^{1+\sigma}(\Omega)$. In consequence,

$$E(t) \leq C \left(\int_\Omega \left(|\nabla\mathbf{u}|^2 + |\mathbf{u}|^{1+\sigma}\right) d\mathbf{x}\right)^\mu, \quad C = C(N,\sigma), \tag{5.4.41}$$

where

$$\mu = \theta + 2\frac{1-\theta}{1+\sigma} = 1 + \frac{2(1-\sigma)}{N(1-\sigma) + 2(1+\sigma)} > 1 \quad (0 < \sigma < 1) \tag{5.4.42}$$

and, from (5.4.39), we obtain the ordinary differential inequality

$$\frac{d}{dt}E(t) + C\,(E(t))^{\frac{1}{\mu}} \leq 0, \quad C = C(N,\delta,\nu,\sigma),$$

whose explicit integration between 0 and $t > 0$ implies $E(t) = 0$, and consequently (5.4.38) holds, where

$$t^* = C^{-1}\frac{4 + N(1-\sigma)}{2(1-\sigma)}E(0)^{\frac{2(1-\sigma)}{4+N(1-\sigma)}}, \tag{5.4.43}$$

concluding the proof.□

Remark 5.4.1 *Analogous remarks as those made for Theorem 5.3.1 can be carried out for this case. For instance, if we consider non-zero* $\mathbf{u}_*$ *and* $\mathbf{u}_\infty$, *this localization effect only occurs, if we assume furthermore the existence of a* $t_* > 0$ *such that* $\mathbf{u}_* = \mathbf{u}_\infty = \mathbf{0}$ *for all* $t \geq t_*$ *and* $E(t_*) < \infty$.

We consider now the case of a bounded subdomain Ω of $\mathbb{R}^N$, $N \geq 3$. Here, instead of (5.4.35)-(5.4.36), we only assume (5.4.35).

Theorem 5.4.2 *Let* $\mathbf{u}$ *be a weak solution of the problem given by* {(5.1.1)-(5.1.3, (5.4.35)} *in* $Q = \Omega \times (0, T)$, *with* $\Omega \subset \mathbb{R}^N$ *bounded,* $N \geq 3$, $0 < T \leq \infty$. *Let* $\mathbf{f}$ *satisfies (5.4.37) and assume* $E(0)$ *is finite. Then, there exists* $t^* \in (0, T)$ *such that*

$$\mathbf{u} = \mathbf{0} \quad \text{for all } t \in [t^*, T) \quad \text{and} \quad \text{for almost all } \mathbf{x} \in \Omega. \tag{5.4.44}$$

Proof: Using a vectorial version of (1.1.6) and proceeding in the same way as we did in the proof of Theorem 5.4.1, we obtain

$$\sqrt{2E(t)} \leq C_1 \|\nabla \mathbf{u}\|^{\theta}_{\mathbf{L}^2(\Omega)} \|\mathbf{u}\|^{1-\theta}_{\mathbf{L}^{1+\sigma}(\Omega)} + C_2 \|\mathbf{u}\|_{\mathbf{L}^{\tilde{q}}(\Omega)} \quad \text{for any } \tilde{q} > 0, \tag{5.4.45}$$

where θ is given in (5.4.40) and $C_i = C_i(\Omega, N, \sigma, \widetilde{q})$, $i = 1, 2$, are positive constants. Because $\mathbf{u} \in \mathrm{L}^2(0, T; \widetilde{\mathbf{H}}_0(\Omega))$, the right-hand side of (5.4.45) is finite if $\|\nabla \mathbf{u}\|_{\mathbf{L}^{1+\sigma}(\Omega)} < \infty$ and $\|\mathbf{u}\|_{\mathbf{L}^{\tilde{q}}(\Omega)} < \infty$ for all $t \in (0, T)$. But, by Hölder's inequality, this is true. Choosing $\tilde{q} = 2/[\theta(\sigma - 1) + 2]$, where θ is given in (5.4.40), and noticing that $0 < \sigma < 1$ implies $(1 + \sigma)/\tilde{q} > 1$, we obtain, after using Hölder's inequality on the second term of the right-hand side, (5.4.41), where now $C = C(N, \sigma, \Omega)$. Then, following in the same way as we did in the proof of Theorem 5.4.1, we prove also that (5.4.44) holds, where t^* is given in (5.4.43), now with $C = C(N, \delta, \nu, \sigma, \Omega)$.□

An interesting situation is when, instead of (5.4.37), we assume the body forces field satisfies, for every $\mathbf{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N$ and for almost all $(\mathbf{x}, t) \in Q$,

$$-\mathbf{f}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{u} \geq \delta |u_i|^{1+\sigma} \quad \text{for some } i \in \{1, \ldots, N\},$$

$\delta > 0$ and $0 < \sigma < 1$. For $N \geq 3$, the localization effect in time, is an open problem, wether $\Omega \subset \mathbb{R}^N$ is bounded or unbounded. The main difficulty relies on the fact that, by one hand, we are not able to show

$$E(t) \leq C \left(\int_{\Omega} \left(|\nabla \mathbf{u}|^2 + |u_i|^{1+\sigma} \right) d\mathbf{x} \right)^{\mu}, \quad C = C(N, \sigma), \quad \mu > 1,$$

and, on the other, we cannot reduce equations (5.1.1)-(5.1.2) to a single one as in the case of $N = 2$.

Chapter 6

Applications to Other Continuum Mechanics Models

> *"Whether we are primarily interested in understanding the physics or in the applications, we must depend heavily on experimental observations to test our analysis and to develop insights into the nature of the phenomenon."*
>
> P.K. Kundu [71, p. 2].

In this chapter we make an attempt to relate our results with some physical applications. As we have said in Section 2.3, our problem has appeared just as purely mathematic and not as any motivation arisen from the applications. Nevertheless, we are now in position to give some possible directions of applications. The possible applications mentioned in this chapter are Classical Elasticity in Section 6.1, Magneto-Hydrodynamics in Section 6.2 and Quasi-Geostrophic Flows in Section 6.3. We are not specialists in these applications, but this chapter shows that this connection should be more deeply studied in a narrow future.

6.1 Classical Elasticity

A fact that arose in Chapter 2 when we have considered the Stokes problem, was the similarity between some results on the exponential decay of the velocity and the Principle of Saint-Venant in classical elasticity (see Remark 2.5.4 on page 41). Thus we wonder if it will be possible to obtain such localization effects for planar problems of Classical Elasticity by considering a forces field such as that one in Chapter 2 satisfying (2.2.9) and (2.2.10). For the sake of comprehension, let us recall the plane problem in Classical Elasticity (cf. Selvadurai [103, §8.1-8.9]). Let us consider a simply connected cross section Ω of a cylindrical body for modelling the two-dimensional elasticity problem. We consider the generalized surface $\partial\Omega$ and we assume that the displacements $\mathbf{u}_0$ are prescribed on a subregion $\partial\Omega_{\mathbf{u}}$ and tractions $\mathbf{T}_0$ are prescribed on another subregion $\partial\Omega_{\mathbf{T}}$. The plane problem in elasticity involves the determination of the displacements $\mathbf{u}$, strains $\mathbf{D} = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and stresses $\mathbf{S} = 2\mu\mathbf{D} + \lambda\mathrm{tr}\mathbf{D}\,\mathbf{I}$ in the elastic medium, with elastic (Lamé) constants λ and μ and body forces $\mathbf{f}$, satisfying

$$\operatorname{div}\mathbf{S} + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{6.1.1}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega_{\mathbf{u}},$$

$$\mathbf{S}\cdot\mathbf{n} = \mathbf{T}_0 \quad \text{on } \partial\Omega_{\mathbf{T}},$$

where $\mathbf{n}$ denotes the outward unit normal to each point of $\partial\Omega_{\mathbf{T}}$. If we are considering a state of plane strain in an elastic medium, with elastic constants λ and μ, and body forces $\mathbf{f}$, which is characterized by displacements $\mathbf{u}$ and stresses $\mathbf{S}$, then the stresses should satisfy the compatibility relation for stresses

$$\operatorname{div}\left[\nabla(\operatorname{tr}\mathbf{S}) + \frac{1}{1-\nu}\mathbf{f}\right] = 0, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{6.1.2}$$

We assume the stress $\mathbf{S}$ can be expressed in terms of the *Airy stress function* ϕ and of the potential V in the forms

$$\mathbf{S}_{11} = \frac{\partial^2\phi}{\partial y^2} - V, \quad \mathbf{S}_{22} = \frac{\partial^2\phi}{\partial x^2} - V, \quad \mathbf{S}_{12} = -\frac{\partial^2\phi}{\partial x\partial y}, \quad \mathbf{f} = \nabla V.$$

Substituting these representations in the equations of static equilibrium (6.1.1), it is evident they are identically satisfied for all choices ϕ and V smooth enough. The stress compatibility relation (6.1.2) gives

$$\triangle^2\phi + \frac{1-2\nu}{1-\nu}\triangle V \equiv \triangle^2\phi + \frac{1-2\nu}{1-\nu}\operatorname{div}\mathbf{f} = 0 \quad \text{in } \Omega.$$

Our point is the following. Is it physically reasonable to consider a forces field $\mathbf{f}$ such that

$$-\mathbf{f}(\mathbf{x},\phi,\nabla\phi)\cdot\nabla\phi \geq \delta|\nabla\phi|^{1+\sigma}$$

for some $\delta > 0$ and $0 < \sigma < 1$? If so, what does it mean in Planar Elasticity? In that case, we can prove, by using the energy method in the same manner as we did for the Stokes problem in Chapter 2, that $\mathbf{S}_{11} = \mathbf{S}_{22} = -V$ and $\mathbf{S}_{12} = \mathbf{S}_{21} = 0$ in an open subdomain of Ω. In consequence, we have $(2\mu+\lambda)\operatorname{div}\mathbf{u} = -2V$ and $\operatorname{curl}\mathbf{u} = 2v_x = -2u_y$ in that domain.

6.2 Magneto-Hydrodynamics

We point out the resemblance between our formulation of Stokes and Navier-Stokes problems presented in Chapters 2 and 3, respectively, and the important question of the confinement of a plasma, typical of magneto-hydrodynamics (MHD). We recall (cf. Freidberg [46] and Landau et al [75]) that MHD concerns with the dynamics of electrically conducting fluids in the presence of magnetic fields. When viscosity and electrical conduction are taken into account and the moving fluid is supposed incompressible and homogeneous, the stationary three-dimensional MHD system involves, among others, the following equations

$$\operatorname{div} \mathbf{v} = 0, \tag{6.2.3}$$

$$-\nu \triangle \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho_0} \left(\mathbf{J} \times \mathbf{B} - \nabla p \right), \tag{6.2.4}$$

$$\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = \frac{1}{\eta} \mathbf{J}. \tag{6.2.5}$$

In these equations, the electromagnetic variables are the *electric field* $\mathbf{E}$, the *magnetic field* $\mathbf{B}$ and the *current density* $\mathbf{J}$. The fluid variables are the fluid velocity $\mathbf{v}$ and the kinematics pressure p. Here, ν is the kinematics viscosity coefficient, ρ_0 is the constant density of the fluid, η stands for the *electrical conductivity coefficient*[1] and c is the *speed of light*. The magnetic field, $\mathbf{B}$, and current density, $\mathbf{J}$, obey Maxwell's equations, which in stationary conditions are

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{and} \quad \operatorname{div} \mathbf{B} = 0, \tag{6.2.6}$$

where μ_0 is the *magnetic permeability*. It is known that the applicability of equations (6.2.3)-(6.2.6) is the smallness of the free path and time of particles (electrons and ions) as compared with characteristics lengths and times (time is considered in the evolutionary case). The goal of MHD equilibrium theory is the discovery of magnetic geometries which, simultaneous confine and isolate hot plasmas from material walls and have good stability properties at sufficient values of β[2], to be promising for use in potential fusion reactors. Usually, the MHD equilibrium is given by equations (6.2.4) and (6.2.6) with the static assumption $\mathbf{v} \equiv \mathbf{0}$. However, stationary equilibria with non-zero flow are possible, but, in this case, there are two main reasons for, in most works in the literature, authors avoid them. The first, is that the kinetic energy of flow represents a source of free energy which often derive instabilities. Second, when plasma flows do occur, they are usually small and are caused by physics not included in the MHD model (see *e.g.* Taylor [114]).

To solve (6.2.3)-(6.2.6), it is necessary to add the adequate boundary conditions. This problem is fully nonlinear and, in the most general case, has no solution. For the case of an ideal MHD, *i.e.* when considering a non-viscous fluid, if the problem

[1] Usually, in Electrodynamics literature, the electrical conductivity is denoted by σ. But here, in order to make no confusion with the constant σ that appears in (2.2.9), we have chosen another Greek letter.

[2] β is the rate of plasma energy over magnetic energy (see Landau et al [75]).

is two-dimensional and the geometry is toroidal axisymmetric[3], the equilibrium equations (6.2.4) and (6.2.6) (with $\mathbf{v} \equiv \mathbf{0}$) can be reduced to the so-called Grad-Shafranov equation

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2} = -r^2\frac{dp}{d\psi} - F\frac{dF}{d\psi},\ p = p(\psi),\ F = F(\psi) = rB_\phi, \tag{6.2.7}$$

where (r, ϕ, z) describe the usual cylindrical coordinate system, B_ϕ is the toroidal field and ψ/r is the toroidal component of the vector potential. For the details, see *e.g.* Freidberg [46, §IV.C] and the references therein to the works by Grad and Rubin (1958), and by Shafranov (1960). Grad-Shafranov equation (6.2.7) is a two-dimensional, nonlinear, elliptic partial differential equation, which in general, must be solved numerically. However, under some assumptions on the explicit form of the pressure profile, the existence and uniqueness of solutions for the Grad-Shafranov equation (6.2.7) can be proved (see *e.g.* Díaz [35] and Mossino [87]).

Now, to see the resemblance between our formulation and the confinement of a plasma, we consider a planar flow ($\mathbf{v} = (\mathbf{u}, 0)$) and the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are given in the form $\mathbf{E}(\mathbf{x}) = (0, 0, E(\mathbf{x}))$ and $\mathbf{B}(\mathbf{x}) = (0, B(\mathbf{x}), 0)$. Then, the equation of motion (6.2.4) becomes

$$-\nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\eta}{\rho_0}(EB, 0) - \frac{\eta}{c\,\rho_0}(B^2u, 0) - \frac{1}{\rho_0}\nabla p.$$

So, the resultant body forces field is a dissipative feedback field given by

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = -\frac{\eta}{\rho_0}(EB, 0) - \frac{\eta}{c\,\rho_0}(B^2u, 0).$$

Nevertheless, we have proved (see Remarks 2.5.4 and 3.3.1) that this field, when $E(\mathbf{x})$ and $B(\mathbf{x})$ are assumed known, is not able to confine the plasma: the x-decay of $\mathbf{u}$ is of exponential type. We wonder, if it is possible to search for electric and magnetic fields satisfying suitable and reasonable physical conditions in order to obtain the localization effect, and, in consequence, the plasma confinement.

6.3 Quasi-Geostrophic Flows

In this section we give some other possible physical application of our results in the study of the quasi-geostrophic equation which appears in Geophysical Fluid Dynamics (GFD). Following Gill [56], Kundu [71] and Pedlosky [93], we recall that the subject of GFD deals with the dynamics of the atmosphere and the ocean. The two features that distinguish GFD from other areas of Fluid Mechanics, are the rotation of the earth and the vertical density stratification of the medium.

[3]The most studied toroidal axisymmetric devices in the literature are Tokamaks (see Gorbunov and Razumova [57]) and Stellarators (see Motley et al [88]). See also Lifschitz [77] for a general discussion of the type of plasma confinement of the Tokamak and the Stellarator.

The rotation of the earth implies that the motion of the atmosphere and the ocean is naturally studied in a coordinate frame rotating with the earth. This gives rise to the *Coriolis*[4] *acceleration*

$$2\boldsymbol{\Omega}\times\mathbf{u} = 2\Omega(w\cos\theta - v\sin\theta, u\sin\theta, -u\cos\theta), \tag{6.3.8}$$

where $\mathbf{u} = (u, v, w)$ is the fluid velocity, $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$ is the angular velocity of the coordinate system, θ is the latitude and Ω is the rate[5] at which earth rotates.

If we are measuring quantities in a rotating frame, we have also to consider the *centrifugal acceleration*

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{R}) = -\Omega^2\mathbf{R},$$

where $\mathbf{R}$ can be taken wether as the vector position of the fluid element or as a vector drawn perpendicular from the axis of rotation to the position of the fluid element.

The equations describing the motion of a fluid rotating with an angular velocity $\boldsymbol{\Omega}$, are given by (1.4.11), (1.4.19) and (1.4.20). Let us drop equations (1.4.11) and (1.4.20), and let us focus our attention on the motion equation (1.4.19), where we have to introduce the effect of earth's rotation. Thus (1.4.19) returns

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\mathbf{x},\mathbf{u}) - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathcal{F}(\mathbf{u}),$$

where $\mathcal{F}(\mathbf{u})$ is the frictional force in the fluid and

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = -2\boldsymbol{\Omega}\times\mathbf{u} + \nabla\Phi. \tag{6.3.9}$$

The first term in (6.3.9) is the Coriolis acceleration (6.3.8) and Φ is the *geopotential*[6] associated with earth's rotation and is given by

$$\nabla\Phi = -\left(\mathbf{g}_N + \Omega^2\mathbf{R}\right), \tag{6.3.10}$$

where $\mathbf{g}_N$ is the acceleration due to (Newtonian) gravitational attractive forces. The direction of $\nabla\Phi$ is called vertical, its magnitude[7] is denoted by g and is called the acceleration due to gravity, and we can write $\Phi = -g\,z$.

Geophysical media are in the form of shallow stratified layers, in which the vertical velocities are much smaller than horizontal velocities, and the stratification of the medium means that the motion has vertical scale small when compared with the scale height. This corresponds to the so-called large-scale motions and which are significantly

[4] When an object is moving in a rotating coordinate system, the path of the object appears to deviate due to the Coriolis effect. If you are in the moving coordinate system, this deviation makes it look like a force is acting upon the object due to Newton's laws of motion. But, actually there is no real force acting on the object, the effect is due to rotation associated with an acceleration of the coordinate system itself.

[5] The earth's rotation rate is $\Omega = 2\pi\,\mathrm{rad/day} = 0.73\times10^{-4}\,\mathrm{s}^{-1}$.

[6] The existence of centrifugal force makes the $\mathbf{g} = \mathbf{g}_N + \Omega^2\mathbf{R}$ less at the equator than at the poles where the centrifugal acceleration is zero (see Kundu [71]).

[7] For most purposes it is sufficiently accurate to take $g = 9.8\,\mathrm{m\,s^{-2}}$.

influenced by the earth's rotation. An important measure of the significance of rotation for a particular phenomenon is the *Rossby number*[8] ϵ.

Then, conform Pedlosky [93, §4.3], the equations governing the motion of a stratified fluid rotating with an angular velocity $\mathbf{\Omega}$, are

$$\mathbf{u}_t - \nu_H \triangle_H \mathbf{u} - \nu_V \triangle_V \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \frac{1}{\rho}\nabla p, \tag{6.3.11}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{6.3.12}$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is given by (6.3.9)-(6.3.10), ν_H and ν_V are the horizontal and vertical turbulent viscosity coefficients, $\triangle_H$ and $\triangle_V$ are the differential operators defined by

$$\triangle_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \triangle_V = \frac{\partial^2}{\partial z^2}.$$

Now, we come back to the Coriolis acceleration (6.3.8), to say that usually only the horizontal components of the vector fields are associated with the Coriolis force. The vertical component of velocity is often neglected, because the thin sheet approximation requires that $w << v$ and consequently we can use $w \cos\theta << v \sin\theta$. Therefore the Coriolis acceleration is given by

$$2\mathbf{\Omega}\times\mathbf{u} = 2\Omega(-v\sin\theta, u\sin\theta, -u\cos\theta). \tag{6.3.13}$$

Moreover, the vertical component of the Coriolis force, namely $-2\Omega u\cos\theta$, is generally negligible compared with the dominant terms in the vertical equation of motion, namely g and p_z. Thus the equation of motion (6.3.11) becomes with the right-hand term $\mathbf{f}(\mathbf{x}, \mathbf{u})$, given in (6.3.9), replaced by

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = -(f\,v, -f\,u, g), \tag{6.3.14}$$

where

$$f = 2\Omega\sin\theta \tag{6.3.15}$$

is the so-called *Coriolis parameter*[9]. This form of the Coriolis force, $2\mathbf{\Omega}\times\mathbf{u} = (f\,v, -f\,u, 0)$, is most relevant in spherical coordinates (cf. Pedlosky [93, §6.2]), but finds itself used in quasi-Cartesian systems approximating a region of the globe as a plane. One of the most used approximations is the so-called *β-plane approximation*, which assumes the variation of f can be approximately represented by expanding f in a Taylor series about the central latitude θ_0

$$f = f_0 + \beta y, \quad f_0 = 2\Omega\sin\theta_0 \quad \text{and} \quad \beta = 2\Omega/R_0\cos\theta_0, \tag{6.3.16}$$

where R_0 is the earth radius[10].

[8] Let L be a characteristic length scale and U a horizontal velocity scale of the motion. The time it takes a fluid element moving with speed U to traverse the distance L is L/U. The Rossby number is defined as the non-dimensional parameter $\epsilon = U/(2\Omega\,L)$ and large-scale flows are defined as those with sufficiently large L such that $\epsilon \leq 1$.

[9] Sometimes it is also called the   or the   .

[10] The radius of earth is nearly $R_0 = 6371\,\mathrm{Km}$.

Geostrophic flows correspond to quasi-steady large-scale motions in the atmosphere or the ocean, away from boundaries. For these flows an excellent approximation for the horizontal equilibrium is the following balance between the Coriolis force and the pressure gradient

$$-f\,v = -\frac{1}{\rho}p_x, \quad f\,u = -\frac{1}{\rho}p_y, \quad g = -\frac{1}{\rho}p_z. \tag{6.3.17}$$

This is the case of flows with both small *Ekman*[11] and Rossby numbers, so that the neglect of nonlinear and viscous terms is justified. The first two equations in (6.3.17) are called the *geostrophic approximation* and the third is the *hydrostatic approximation.*

Quasi-geostrophic flows are nearly geostrophic flows in which the time dependent forces are much smaller than the pressure and Coriolis forces in the horizontal plane. We notice that the geostrophic approximation about which most of the mathematical theory is developed, fails near the equator, within a latitude belt of $\pm 3^{\circ}$, where the Coriolis force on horizontal currents is extremely feeble. Hence by its nature quasi-geostrophic theory must be less than global.

Following Pedlosky [93, §4.5, §4.11, §6.2], the equations of motion {(6.3.11)-(6.3.12), (6.3.14)-(6.3.15)} are scaled in terms of the characteristic vertical and horizontal scales, say D and L respectively. Then $\mathbf{u}$ and p are expanded in Taylor series about a small Rossby number ϵ. The pressure gradient is eliminated by cross differentiation with respect to x and y, and it is assumed the concept of the β-plane (see (6.3.16)). Finally, we obtain the so-called quasi-geostrophic potential vorticity equation with friction and topography, which is written in terms of the geostrophic stream function $\psi(x, y, t)$

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)(\triangle\psi - F\psi + \beta y) = \beta \mathbf{k}\cdot \mathbf{curl}\tau - \frac{r}{2}\triangle\psi + \frac{1}{\mathcal{R}}\triangle^2\psi. \tag{6.3.18}$$

Here $\beta = 2\Omega\, L^2/(R_0\, U)\cos\theta$ is sometimes called the meridional gradient of the Coriolis parameter, $\mathcal{R} = 2\epsilon/E_H$ is the Reynolds number which in most cases of geophysical interest is quite large and so $\mathcal{R}^{-1}$ is a small parameter, $F = f^2 L^2/(gD) = (L/R)^2$, R is the *Rossby deformation radius*[12] for the layer of depth D, $r = \sqrt{E_V}/\epsilon$ is sometimes called the *Ekman dissipation constant*, $E_V = 2\nu_V/(fD^2)$ and $E_H = 2\nu_H/(fD^2)$ are the *vertical* and *horizontal Ekman numbers*, $\mathbf{k}$ is a unit vector in the z-direction, τ is the *Ekman layer friction* corresponding to the stress exerted by a rigid surface on the fluid (cf. Pedlosky [93, §4.3]), or the *Ekman pumping velocity* corresponding to the wind stress (cf. Gill [56, §9.4]).

Problem consisting of equation (6.3.18) together with appropriated initial and boundary conditions has attracted the interest of many mathematicians in the last years in order to study its well-posedness. Most works on this fluid model are for the case $F = 0$, *i.e.* it is assumed that $R >> L$ which from the point of view of vorticity balance, the free surface appears no difference than a rigid lid. See *e.g.* the works by Brannan et al [25], Gao and Duan [54], and J-L Lions et al [81].

[11] The Ekman number is the ratio of the frictional force per unit mass to the Coriolis acceleration: $E = \nu/(2\Omega\, L^2)$, where ν is the kinematics viscosity coefficient.

[12] $R = \sqrt{gD}/f$ is the scale for which the relative vorticity and the surface height (vortex-tube stretching) make equal contributions to the potential vorticity.

The linkage that we would like to make between our results and this fluid model is the following. It could be interesting and physically reasonable to look at the influence of the Coriolis force on the stopping distance of a quasi-geostrophic flow.

Chapter 7

Conclusions

> *"Once started the research in mathematics and if we want, we have work for our lifetime."*
>
> S.N. Antontsev.

As we begun by saying at the Preface, this work started with the research of a suitable forces field which could be able to stop a fluid, governed by the incompressible Navier-Stokes equations in a semi-infinite strip with appropriated initial and or boundary conditions, at a finite distance from the strip entrance. Through this text, we have seen that a feedback forces field satisfying suitable nonlinear conditions on the first component and suitable dissipative conditions on the second is able to stop a fluid governed by the incompressible stationary Stokes and Navier-Stokes equations and by the stationary Boussinesq approximation as well. Although it is not completely solved the time dependent incompressible Navier-Stokes problem, we already have proven for this case the finite speed of propagations property. Moreover, though we have not proven, one can consider the problem of time dependent Boussinesq approximation and prove the finite speed of propagations by using the results already proved for the time dependent Navier-Stokes problem. We hope to establish the same localization effects in space for these time dependent cases very soon. On the other hand, one can realize that by using the same techniques, it will be possible, with more or less difficulty, to establish the same localization effects for other fluid flow models. One of these models that we will study next, is the case of an incompressible non-Newtonian fluids where the stress tensor satisfies, for instance, the Ostwald-de Waele laws (1.4.25)-(1.4.27). Many other models can be studied, for instance, all the cases mentioned above but for compressible fluids.

It may be considered many other reasonable planar shape domains from the point of view of Physics and try to prove the same localization effects. During this work we have mainly considered horizontal or vertical strips wether constant or not.

We may also wonder what happen if we consider different boundary conditions. For instance, if we consider a semi-infinite strip with horizontal walls but where in some finite part of the horizontal walls there is no fluid adherence. Possibly in that finite part we may have some inflow or outflow conditions. In that conditions, would we have the same localization effects?

There is also the question to treat these problems with Numerical Analysis. One of the interest is to find better estimates and in consequence minimal distances from the strip entrance from where on the fluid is in rest. There is also the interest in develop some numerical implementations to see the behavior of the streamlines in all the cases that the localizations effects hold. Our intuition says that when a spatial localization effect occurs, the streamlines must be described as in Figures 7.1 or 7.2.

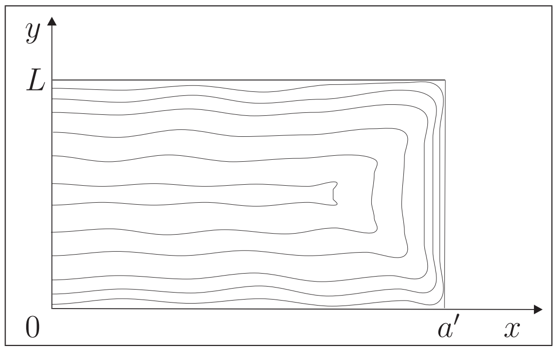

Figure 7.1: Horizontal streamlines.

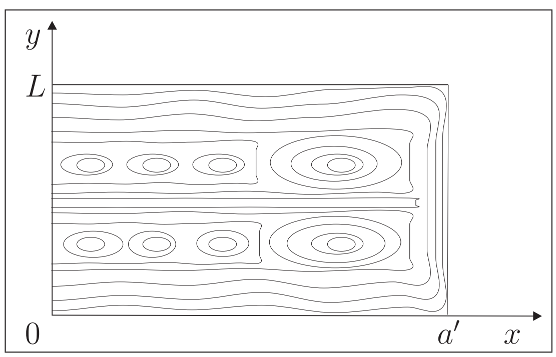

Figure 7.2: Circular streamlines.

In Chapter 6, we made reference to some possible applications of our results. As we said before, though we are working in the context of Fluid Mechanics, the forces field we have considered are rather purely mathematic than any motivation arisen from Continuum Mechanics. But, in that chapter, we have made reference to three different areas of Continuum Mechanics where our results may be applied. Many other application of these problems can be possible. For instance, the localization effects for the problem of Boussinesq approximation, where the stationary case was studied in Chapter 7, may show how the action on two simultaneous effects, a suitable coupling between velocity and temperature and a low range of temperature but upper than the phase changing one, may be responsible of stopping a viscous fluid without any changing phase. This philosophy could be useful in the monitoring of many flows problems, especially in metallurgy. Other interesting and possible application are blood flows. One can see that blood vessels can be modelled by strip shape domains and it could be interesting for medical purposes to study the forces fields which can stop blood flows.

An interesting situation, from the purely mathematical point of view and also from applications, is when the power σ in assumptions (2.2.9), (4.2.8) and (5.1.7) may also depend on the spatial variable $\mathbf{x}$, and also on t for the time dependent cases. In the last decade we witness a strong rise of interest in the study of various mathematical problems in the so-called spaces with non-standard growth. This expression mainly relates to the generalized Lebesgue spaces $\mathrm{L}^{p(\cdot)}(\Omega)$, with variable order $p(x)$, and to the corresponding generalized Sobolev spaces $\mathrm{W}^{k,p(\cdot)}(\Omega)$. This theory is far from being complete in the sense that for the Lebesgue and Sobolev spaces with fixed exponents is. But there are now many results which allow us to prove certain properties of our fluid problems. See *e.g.* the survey by Samko [100] for an explanation of this theory and the monograph by Růžička [97] where this theory is applied to *electro-rheological fluids*[1].

As one can see from the foregoing paragraphs, there is a lot of problems that know are waiting for being solved. But not underestimating all these problems, because most of them are not easy tasks, our main goal is the three-dimensional incompressible Navier-Stokes problem. As we have mentioned, in the final part of Chapter 5, the main difficulty here in obtaining the localization effect in time, is because we cannot reduce equation of motion (1.4.28) to a single one as for the two-dimensional case. In the three-dimensional case, we only can reduce (1.4.28) to a system of equations where the vorticity and velocity are coupled. For the classical incompressible Navier-Stokes equations, *i.e.* with prescribed linear forces field, there are some results on the localization of the vorticity and its applications to the asymptotic behavior of solutions (see *e.g.* Brandolese [24], and Gallay and Wayne [53]). Using their analysis and the energy methods as we did, possibly we will be able to find a suitable forces field (with three components) which can stops a fluid, governed by the Navier-Stokes equations, at a finite distance from the entrance of the three-dimensional domain, a cylinder for instance.

The resolution of the problem stated in the precedent paragraph may open some windows in the direction of solving the three-dimensional incompressible Navier-Stokes problem in the large. It is known since the works by Leray [76, pp. 18-159], Hopf [61], J-L Lions and Prodi [80], and Ladyzhenskaya [72] that in dimension two the theory is fairly satisfactory, the problem is well-posed in the sense of Hadamard[2] [59], but for dimension three we have only partial results. In this case we have existence and uniqueness of a strong solution in some interval $(0, T^*)$, with T^* depending on the data, existence of weak solutions in the interval $(0, \infty)$. But uniqueness of weak solutions is still an open problem, as well the existence for all time of strong solutions. Many of the best specialists in this field, as O. Ladyzhenskaya [73] and R. Temam [116], suggest the possible occurrence of singularities in the three-dimensional incompressible Navier-Stokes equations and thus $\|\mathbf{u}(t)\|$ becomes infinite in finite time. The problem of existence of such singularities has not been proved nor disproved, in spite of many

[1]Electro-rheological fluids are special viscous fluids, which are characterized by their ability to undergo significant changes in their mechanical properties due to the application of an electric field.

[2]See page 7.

attempts that have been made. In fact, this is one of the Millennium Prizes Problem attributed by the Clay Mathematics Institute[3] to whom can be able to solve it. One attempt to solve this problem, was made by Sheffer [106] who considered weak solutions to the time dependent Navier-Stokes problem of incompressible fluid flow in three-dimensional with an external forces field that always acts against to the flow. This forces field was characterized by

$$\mathbf{f} \cdot \mathbf{u} \leq 0 \text{ for every } \mathbf{u} \in \mathbb{R}^3 \quad \text{and} \quad \operatorname{div} \mathbf{f} = 0.$$

Sheffer's attempt was to prove the conjecture claiming that, with a forces field like this, there is a solution to the Navier-Stokes equations with an internal singularity. So far, Sheffer only made a few steps towards to prove this conjecture. Our point concerning this challenger problem is the following. Imagine that we could prove the extinction in a finite time of the solutions of the three dimensional incompressible Navier-Stokes equations with suitable forces field as those we have considered through this text. Would that be enough to solve this Navier-Stokes Millennium Problem in the sense of disprove the existence of internal singularities and consequently to prove the existence of a strong solution in the interval $(0, \infty)$?

[3] See .

Appendix

A. Notation

The notation used foregoing is largely standard in Analysis and, in particular, in Partial Differential Equations and in Functional Analysis. We also have used specific notation from Fluid Mechanics concerning Tensor Analysis. For many of this notation, we have followed the monograph by Evans [44].

We use usual notation for the basic numbers sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are the sets of natural, integer, rational and real numbers, respectively. $\mathbb{R}^N$ is the N-dimensional euclidian space, $N \in \mathbb{N}$ and for $N = 1$ we set $\mathbb{R}^1 = \mathbb{R}$. The points of $\mathbb{R}^N$ are denoted by $\mathbf{x} = (x_1, \dots, x_N)$ and corresponding vectors by $\mathbf{u} = (u_1, \dots, u_N)$. We use the notations $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u, v, w)$ for $N = 3$, $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$ for $N = 2$, and x and u for $N = 1$. The euclidian or scalar product in $\mathbb{R}^N$ is denoted by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{N} x_i y_i$ and the euclidian norm by $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. The angle formed by two vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^N$, is denoted by $\angle(\mathbf{u}, \mathbf{v})$. The cross product between two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by $\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$.

In our considerations, Ω always denotes a subdomain of $\mathbb{R}^N$, $N \geq 1$, *i.e.* a simply connected subset of $\mathbb{R}^N$, not necessarily bounded, and ω a subdomain of Ω in the same sense. For any $\Omega \subset \mathbb{R}^N$, we denote by $\partial\Omega$ its compact boundary, by $\overline{\Omega}$ its closure, *i.e.* $\Omega \cup \partial\Omega$, and we say Ω is compact, if $\overline{\Omega} = \Omega$. Given $\omega \subset \Omega$, we write $\omega \subset\subset \Omega$ if $\overline{\omega} \subset \Omega$. Particular importance is given to the following subsets of the planar domain Ω resulting from the non-empty intersection of Ω with the half-planes $\{(x, y) \in \mathbb{R}^2 : x > a\}$ and $\{(x, y) \in \mathbb{R}^2 : x < a\}$, with $a \geq 0$, respectively

$$\Omega_a = \{(x, y) \in \Omega : x > a\} \quad \text{and} \quad \Omega^a = \{(x, y) \in \Omega : x < a\}.$$

For any measurable subset E of $\mathbb{R}^N$, we denote by $|\mathrm{E}|$ the Lebesgue measure of E.

Let $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ and $N \geq 1$, be a function with sufficient regularity. The support of u is denoted by $\operatorname{supp} u$ and is defined by

$$\operatorname{supp} u = \overline{\{\mathbf{x} \in \Omega : u(\mathbf{x}) \neq 0\}}.$$

The symbols u_+ and u_- are used with the following meaning

$$u_+ = \max(u, 0) \quad \text{and} \quad u_- = \min(u, 0).$$

For $1 \leq i \leq N$, we use $\frac{\partial u}{x_i}$ to denote the partial derivative of u in order to the i-th component of $\mathbf{x}$. We usually will write u_{x_i} for $\frac{\partial u}{x_i}$. Similarly $\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j}$, $\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = u_{x_i x_j x_k}$, *etc.* If $N = 1$, the derivative of u is denoted by $\frac{d}{dx}u$ or simply by using superscript primes as $u'(x)$. In general, if k is a non-negative integer, we write $\mathrm{D}^k u = \{\mathrm{D}^\alpha u : |\alpha| = k\}$ for the set of all partial derivatives of order k, where

$$\mathrm{D}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_N,$$

with $\alpha = (\alpha_1, \ldots, \alpha_N)$ a multi-index, *i.e.* α_i, $1 \leq i \leq N$, are non-negative integers, and

$$\left|\mathrm{D}^k u\right| = \sqrt{\sum_{|\alpha|=k} |\mathrm{D}^\alpha u|^2}$$

If $k = 1$, we regard the elements of $\mathrm{D}u$ as being arranged in a vector, called gradient vector and often denoted by

$$\nabla u = (u_{x_1}, \ldots, u_{x_N}).$$

If $k = 2$, the elements of $\mathrm{D}^2 u$ are regard as being arranged in a $N \times N$ matrix, called Hessian matrix, and defined by

$$\mathrm{D}^2 u = \left[u_{x_i x_j}\right], \quad i,\, j = 1, \ldots, N.$$

We define the Laplacian of u as

$$\triangle u = \sum_{i=1}^{N} u_{x_i x_i} = \mathrm{tr}\left(\mathrm{D}^2 u\right)$$

and, in general, for any positive integer p, the p-Laplacian of u is defined by $\triangle^p u = \triangle(\triangle^{p-1} u)$.

We distinguish vector-valued functions from the above ones by using bold letters. Let M, $N > 1$ (the case $M = 1$ is aforementioned and $N = 1$ is trivial), $\Omega \subset \mathbb{R}^N$ and $\mathbf{u} : \Omega \to \mathbb{R}^M$, $\mathbf{u} = (u_1, \ldots, u_M)$, be a function with sufficient regularity. We define $\mathrm{D}^\alpha \mathbf{u} = (\mathrm{D}^\alpha u_1, \ldots, \mathrm{D}^\alpha u_M)$ for each multi-index α. Then $\mathrm{D}^k \mathbf{u} = \{\mathrm{D}^\alpha \mathbf{u} : |\alpha| = k\}$ and

$$|\mathrm{D}^k \mathbf{u}| = \sqrt{\sum_{|\alpha|=k} |\mathrm{D}^\alpha \mathbf{u}|^2},$$

as before. In the special case of $k = 1$, we write

$$\mathrm{D}\mathbf{u} = \nabla \mathbf{u} = \left[\frac{\partial u_i}{\partial x_j}\right], \quad 1 \leq i \leq M,\ 1 \leq j \leq N,$$

for the gradient matrix. If $M = N$, we define the divergence of $\mathbf{u}$ by

$$\mathrm{div}\, \mathbf{u} = \mathrm{tr}\,(\nabla \mathbf{u}) = \sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i},$$

and the Laplacian of $\mathbf{u}$ by

$$\triangle\mathbf{u} = (\triangle u_1, \ldots, \triangle u_N)\,.$$

We use the notations $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $\mathbf{u}\cdot\nabla\mathbf{u}\cdot\varphi$ with the following meaning

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \sum_{i=1}^{N} u_i \mathbf{u}_{x_i} = \left(\sum_{i=1}^{N} u_i \frac{\partial u_1}{\partial x_i}, \ldots, \sum_{i=1}^{N} u_i \frac{\partial u_N}{\partial x_i}\right),$$

$$\mathbf{u}\cdot\nabla\mathbf{u}\cdot\varphi = (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\varphi = \sum_{j=1}^{N}\sum_{i=1}^{N} u_i \frac{\partial u_j}{\partial x_i}\varphi_j.$$

In the special case of $M = N = 3$, we define the vector $\mathbf{curl}\,\mathbf{u}$, (recall that $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u, v, w)$) as

$$\mathbf{curl}\,\mathbf{u} = (w_y - v_z, u_z - w_x, v_x - u_y).$$

If $M = N = 2$, the vector $\mathbf{curl}\,\mathbf{u}$ (recall that $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$) has only one component, orthogonal to $\mathbf{u}$, thus can be consider as a scalar and is defined by

$$\mathrm{curl}\,\mathbf{u} = v_x - u_y.$$

Tensors are matrix-valued functions of order $N \times N$ and are represented by capital bold letters. In Continuum Mechanics we are only concerned with the case $N = 3$ or $N = 2$, but the following notations can easily be extended for an arbitrary N. Notice that for a vector-valued function $\mathbf{u} : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$, the matrix gradient $\nabla\,\mathbf{u}$ is a tensor in this sense. We consider the case $N = 3$, for $N = 2$ the notations are analogous. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, $i,\ j = 1, 2, 3$, be two tensors fields where their components a_{ij} and b_{ij} are scalar-valued functions defined in a domain $\Omega \subset \mathbb{R}^3$. We define the transpose tensor of $\mathbf{A}$ by $\mathbf{A}^T = [a_{ji}]$, the product of two tensors $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A}\,\mathbf{B} = \left[\sum_{k=1}^{3} a_{ik}b_{kj}\right]$, $i,\ j = 1, 2, 3$, and the trace of $\mathbf{A}$ by $\mathrm{tr}\,\mathbf{A} = \sum_{i=1}^{3} a_{ii}$. The convolution product of two tensors is defined by

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}\,(\mathbf{A}\,\mathbf{B}^T) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij}$$

and the modulus of $\mathbf{A}$ by $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}$. The divergence of a tensor $\mathbf{A}$ is a vector-valued function and is defined by

$$\mathbf{div}\,\mathbf{A} = (\mathrm{div}\,\mathbf{a_1}, \mathrm{div}\,\mathbf{a_2}, \mathrm{div}\,\mathbf{a_3})\,, \quad \mathbf{a_i} = (a_{i1}, a_{i2}, a_{i3}),\ i = 1, 2, 3.$$

We define the principal invariants of a tensor $\mathbf{A}$ by

$$I_1 = \mathrm{tr}(\mathbf{A}), \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right], \quad I_3 = \det\mathbf{A}.$$

More generally, an invariant of a matrix $\mathbf{A}$ is any real-valued function $f(\mathbf{A})$ with the property that $f(\mathbf{A}) = f(\mathbf{B}^{-1}\mathbf{A}\mathbf{B})$ for all invertible matrices $\mathbf{B}$.

B. Function Spaces

In this section we introduce the functional spaces that are used in this text. The presentation is essentially based on the monographs by Adams [1], Kufner et al [70], Maz'ja [85] and Nečas [90].

Given a non-negative integer k and a subdomain $\Omega \subset \mathbb{R}^N$, we define, for any integer $N \geq 1$, the linear space $\mathrm{C}^k(\Omega)$ as the space of all functions $u : \Omega \to \mathbb{R}$ such that $\mathrm{D}^\alpha u$ is continuous in Ω for all $0 \leq |\alpha| \leq k$. We set $\mathrm{C}^0(\Omega) = \mathrm{C}(\Omega)$ and $\mathrm{C}^\infty(\Omega) = \cap_{k=0}^\infty \mathrm{C}^k(\Omega)$. The symbols $\mathrm{C}_0^k(\Omega)$ and $\mathrm{C}_0^\infty(\Omega)$ indicate the linear subspaces of $\mathrm{C}^k(\Omega)$ and $\mathrm{C}^\infty(\Omega)$, respectively, of all those functions which have compact support in Ω. We define $\mathrm{C}^k(\overline{\Omega})$ as the space of all functions u in $\mathrm{C}^k(\Omega)$ such that $\mathrm{D}^\alpha u$ is bounded and uniformly continuous in Ω for all $0 \leq |\alpha| \leq k$. $\mathrm{C}^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{\mathrm{C}^k(\overline{\Omega})} = \max_{0 \leq |\alpha| \leq k} \sup_{\mathbf{x} \in \Omega} |\mathrm{D}^\alpha u(\mathbf{x})|.$$

Again, we set $\mathrm{C}^0(\overline{\Omega}) = \mathrm{C}(\overline{\Omega})$ and $\mathrm{C}^\infty(\overline{\Omega}) = \cap_{k=0}^\infty \mathrm{C}^k(\overline{\Omega})$.

For $0 < \lambda \leq 1$, we define the space $\mathrm{C}^{k,\lambda}(\overline{\Omega})$ as the subset of all functions $u \in \mathrm{C}^k(\overline{\Omega})$ such that

$$\sup_{\mathbf{x},\, \mathbf{y} \in \Omega,\ \mathbf{x} \neq \mathbf{y}} \frac{|\mathrm{D}^\alpha u(\mathbf{x}) - \mathrm{D}^\alpha u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\lambda} < \infty$$

for all α with $|\alpha| = k$. $\mathrm{C}^{k,\lambda}(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{\mathrm{C}^{k,\lambda}(\overline{\Omega})} = \|u\|_{\mathrm{C}^k(\overline{\Omega})} + [u]_{k,\lambda}, \quad [u]_{k,\lambda} = \max_{0 \leq |\alpha| \leq k} \sup_{\mathbf{x},\, \mathbf{y} \in \Omega,\ \mathbf{x} \neq \mathbf{y}} \frac{|\mathrm{D}^\alpha u(\mathbf{x}) - \mathrm{D}^\alpha u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\lambda}.$$

We set $\mathrm{C}^{0,\lambda}(\overline{\Omega}) = \mathrm{C}^\lambda(\overline{\Omega})$ and the functions in $\mathrm{C}^\lambda(\overline{\Omega})$ are said *Hölder continuous*, and *Lipschtitz continuous* if $\lambda = 1$.

For $1 \leq p \leq \infty$ and a subdomain $\Omega \subset \mathbb{R}^N$, $\mathrm{L}^p(\Omega)$ denotes the linear space of all (equivalent classes of) real Lebesgue-measurable functions u defined in Ω such that

$$\int_\Omega |u|^p \, d\mathbf{x} < \infty \quad \text{if} \quad 1 \leq p < \infty,$$

or

$$\text{ess sup}_\Omega |u| = \inf\{\alpha : |u(\mathbf{x})| \leq \alpha \text{ a.e. in } \Omega\} < \infty \quad \text{if} \quad p = \infty.$$

$\mathrm{L}^p(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{\mathrm{L}^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \, d\mathbf{x}\right)^{1/p} & \text{if } 1 < p < \infty \\ \text{ess sup}_\Omega |u| & \text{if } p = \infty. \end{cases}$$

These spaces are usually denoted as the *Lebesgue spaces*. If $p = 2$, $\mathrm{L}^p(\Omega)$ is a Hilbert space under the scalar product

$$(u, v) = \int_\Omega u\, v \, d\mathbf{x}.$$

We write $u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ to mean $u \in \mathrm{L}^p(\omega)$ for any $\omega \subset\subset \Omega$. If $1 \le p < \infty$, then $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}^p(\Omega)$ and $\mathrm{L}^p(\Omega)$ is a separable space. For $1 \le p \le \infty$, we denote by p' the conjugate exponent of p, *i.e.* $1/p + 1/p' = 1$, where it is understood that if $p = 1$ then $p' = \infty$ and reciprocally. The dual space of $\mathrm{L}^p(\Omega)$ is (isometrically isomorphic[4] to) $\mathrm{L}^{p'}(\Omega)$ if $1 \le p < \infty$ and the dual space of $\mathrm{L}^\infty(\Omega)$ contains strictly $\mathrm{L}^1(\Omega)$. The duality between $\mathrm{L}^p(\Omega)$ and $\mathrm{L}^{p'}(\Omega)$ becomes

$$\langle u, v\rangle_{\mathrm{L}^{p'}(\Omega)\times\mathrm{L}^p(\Omega)} = \int_\Omega u\, v\, d\mathbf{x} \quad \text{for } u \in \mathrm{L}^{p'}(\Omega),\ v \in \mathrm{L}^p(\Omega).$$

Let $1 \le p < \infty$, then we say that u_n converges weakly to u, in $\mathrm{L}^p(\Omega)$, as n tends to infinity, if

$$\int_\Omega u_n\, v\, d\mathbf{x} \to \int_\Omega u\, v\, d\mathbf{x}, \text{ as } n \to \infty \quad \text{for every } v \in \mathrm{L}^{p'}(\Omega).$$

If $p = \infty$, we say that u_n converges weak-star to u, in $\mathrm{L}^\infty(\Omega)$, as n tends to infinity, if

$$\int_\Omega u_n\, v\, d\mathbf{x} \to \int_\Omega u\, v\, d\mathbf{x}, \text{ as } n \to \infty \quad \text{for every } v \in \mathrm{L}^1(\Omega).$$

$\mathrm{L}^p(\Omega)$ is a reflexive space if $1 < p < \infty$ and, in that case[5], every bounded sequence in $\mathrm{L}^p(\Omega)$ has a weakly convergent subsequence to an element of $\mathrm{L}^p(\Omega)$.

Following the notation of the Theory of Distributions, let us set $\mathcal{D}(\Omega) = \mathrm{C}_0^\infty(\Omega)$ and consider its dual space $\mathcal{D}'(\Omega)$, called the *Schwarz space* or the *space of distributions*. Given $u \in \mathcal{D}'(\Omega)$, we say that v_i is the i-th partial generalized derivative of u, or in the sense of distributions, if

$$\langle u, \varphi_{x_i}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = -\langle v_i, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}, \quad \text{for every } \varphi \in \mathcal{D}(\Omega).$$

If $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, the Riesz Representation Theorem, allows us to write

$$\langle u, \varphi_{x_i}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega u\varphi_{x_i}\, d\mathbf{x}.$$

Then, if the above v_i exists and is in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we say that v_i is the i-th partial weak derivative of u, which by the convenience of notation we denote by u_{x_i}, and

$$\int_\Omega u\, \varphi_{x_i}\, d\mathbf{x} = -\int_\Omega u_{x_i}\, \varphi\, d\mathbf{x}, \quad \text{for every } \varphi \in \mathcal{D}(\Omega).$$

Analogously, for any multi-index α, we say that $\mathrm{D}^\alpha u$ is the α-th partial weak derivative of $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, if

$$\int_\Omega u\, \mathrm{D}^\alpha\varphi\, d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega \mathrm{D}^\alpha u\, \varphi\, d\mathbf{x}, \quad \text{for every } \varphi \in \mathcal{D}(\Omega).$$

[4]Two normed spaces X and Y with norms $\|\cdot\|_{\mathrm{X}}$ and $\|\cdot\|_{\mathrm{Y}}$, respectively, are isometrically isomorphic if there exists a one-to-one linear operator L from X onto Y such that $\|L(x)\|_{\mathrm{Y}} = \|x\|_{\mathrm{X}}$ for all $x \in \mathrm{X}$.

[5]If X is a reflexive Banach space and x_n a sequence in X such that $\|x_n\|_X \le M$ for $M > 0$ constant, then, there exist $x \in X$ and a subsequence x_{n_k} such that $x_{n_k} \to x$ weakly in X.

For $1 \leq p \leq \infty$, any non-negative integer k and a domain $\Omega \subset \mathbb{R}^N$, $\mathrm{W}^{k,p}(\Omega)$ denotes the linear space of all functions $u \in \mathrm{L}^p(\Omega)$ such that the weak derivatives $\mathrm{D}^\alpha u$ exist and are in $\mathrm{L}^p(\Omega)$ for any multi-index α such that $0 \leq |\alpha| \leq k$. These spaces are known in the literature as the *Sobolev spaces* and they are Banach spaces with respect to the norm

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|=0}^k \int_\Omega |\mathrm{D}^\alpha u|^p \, d\mathbf{x}\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sum_{|\alpha|=0}^k \operatorname{ess\,sup}_\Omega |\mathrm{D}^\alpha u| & \text{if } p = \infty \end{cases}$$

When $k = 0$, we set $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$. If Ω is bounded, then $\mathrm{W}^{1,\infty}(\Omega)$ is the space of Lipschitz functions. For any non-negative integer k, $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ is defined analogously to $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$. If $p = 2$, $\mathrm{W}^{k,2}(\Omega)$ is a Hilbert space under the scalar product

$$(u, v) = \sum_{|\alpha|=0}^{k} \int_\Omega \mathrm{D}^\alpha u \, \mathrm{D}^\alpha v \, d\mathbf{x}$$

and is usually denoted by $\mathrm{H}^k(\Omega)$. For any $1 \leq p < \infty$ and $k \in \mathbb{N}$, we define $\mathrm{W}^{k,p}_0(\Omega)$ as the closure of $\mathrm{C}^\infty_0(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$ and if $p = 2$, we denote this space by $\mathrm{H}^k_0(\Omega)$. For $1 \leq p < \infty$, the dual space of $\mathrm{W}^{k,p}(\Omega)$ is denoted by $\left(\mathrm{W}^{k,p}(\Omega)\right)'$ and the duality between these two spaces is defined by

$$\langle u, v \rangle_{\left(\mathrm{W}^{k,p}(\Omega)\right)' \times \mathrm{W}^{k,p}(\Omega)} = \langle w, v \rangle_{\mathrm{L}^{p'}(\Omega) \times \mathrm{L}^p(\Omega)} + \sum_{|\alpha|=k} \langle w_\alpha, \mathrm{D}^\alpha v \rangle_{\mathrm{L}^{p'}(\Omega) \times \mathrm{L}^p(\Omega)}$$

for every $v \in \mathrm{W}^{k,p}(\Omega)$, where w and w_α, with α varying and such that $|\alpha| = k$, are elements of $\mathrm{L}^{p'}(\Omega)$ corresponding to a given u in $\left(\mathrm{W}^{k,p}(\Omega)\right)'$. If $1 \leq p < \infty$, we set $\left(\mathrm{W}^{k,p}_0(\Omega)\right)' = \mathrm{W}^{-k,p'}(\Omega)$ (up to an isometric isomorphism), with the notation $\mathrm{H}^{-k}(\Omega)$ if $p = 2$. $\mathrm{W}^{k,p}(\Omega)$, as a subspace of $\mathrm{L}^p(\Omega)$, inherits most of the properties of $\mathrm{L}^p(\Omega)$. For any non-negative integer k, it is a separable space if $1 \leq p < \infty$, reflexive if $1 < p < \infty$. Moreover, if $1 \leq p < \infty$, $\mathrm{W}^{k,p}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$.

Now we consider the traces of functions in $\mathrm{W}^{k,p}(\Omega)$. Let Ω be a subdomain of $\mathbb{R}^N$ sufficiently regular, say locally Lipschitz. If $kp > N$, every function in $\mathrm{W}^{k,p}(\Omega)$ can be redefined on a set of zero measure in such a way that it becomes, at least, continuous up to the (compact) boundary. However, if $kp \leq N$, this is no longer valid. Nevertheless it is possible to define, in a suitable sense, the trace of a function defined in any Sobolev space $\mathrm{W}^{k,p}(\Omega)$. For our purposes, we will only consider the case $k = 1$, and for an arbitrary k, we address the reader for the monographs by Kufner et al [70] and Nečas [90]. If $1 \leq p < N$ and $q = (Np-p)/(N-p)$ or $N \leq p < \infty$ and $1 \leq q < \infty$, there exists a uniquely determined linear mapping, called trace, $T : \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^q(\partial\Omega)$ such that $T(u) = u_{|\partial\Omega}$ for all $u \in \mathrm{C}^\infty(\overline{\Omega})$. For $1 \leq p < \infty$, we denote by $\mathrm{W}^{1-1/p,p}(\partial\Omega)$ the subspace of all functions $u \in \mathrm{L}^p(\partial\Omega)$ such that

$$\langle\langle u \rangle\rangle_{\mathrm{W}^{1-1/p,p}(\partial\Omega)} = \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(y) - u(x)|^p}{|x - y|^{N-2+p}} \, d\mathbf{s}_y \, d\mathbf{s}_x < \infty.$$

$W^{1-1/p,p}(\partial\Omega)$ is a Banach space for the norm

$$\|u\|_{W^{1-1/p,p}(\partial\Omega)} = \|u\|_{L^p(\partial\Omega)} + \langle\langle u\rangle\rangle_{W^{1-1/p,p}(\partial\Omega)}$$

and is a dense subset of $L^p(\partial\Omega)$. Furthermore, it is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. For Ω sufficiently regular $W^{1-1/p,p}(\partial\Omega) \cap C^\infty(\partial\Omega)$ is dense in $W^{1-1/p,p}(\partial\Omega)$. If $p = 2$, $W^{1/2,2}(\partial\Omega)$ is a Hilbert space under the scalar product

$$(u, v) = \int_{\partial\Omega} u\, v\, d\mathbf{s} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{(u(y) - u(x))\,(v(y) - v(x))}{|x - y|^N}\, d\mathbf{s}_y\, d\mathbf{s}_x,$$

and we denote this space by $H^{1/2}(\partial\Omega)$.

$W_0^{1,p}(\Omega)$ can be characterized as the space of functions in $W^{1,p}(\Omega)$ having zero trace on $\partial\Omega$. In this way, for any $k \in \mathbb{N}$, we can interpret $W_0^{k,p}(\Omega)$ as comprising those functions $u \in W^{k,p}(\Omega)$ such that $D^\alpha u$ has zero trace on $\partial\Omega$ for all $0 \leq |\alpha| \leq k - 1$.

For a given Banach space X and a real interval (a, b), with $a < b \leq \infty$, we denote by $L^p(a, b; X)$ the linear space of (equivalent class of) Lebesgue-measurable functions $u : (a, b) \to X$ such that

$$\int_a^b \|u(t)\|_X^p\, dt < \infty \quad \text{if} \quad 1 \leq p < \infty,$$

or

$$\text{ess sup}_{(a,b)}\|u(t)\|_X < \infty \quad \text{if} \quad p = \infty.$$

$L^p(a, b; X)$ is usually called the *Bochner space* and is a Banach space endowed with the norm

$$\|u\|_{L^p(a,b;X)} = \begin{cases} \left(\int_a^b \|u(t)\|_X^p\, dt\right)^{1/p} & \text{if } 1 < p < \infty \\ \text{ess sup}_{(a,b)}\|u(t)\|_X & \text{if } p = \infty. \end{cases}$$

For $1 \leq p < \infty$, the dual space of $L^p(a, b; X)$ is (isometrically isomorphic to) $L^{p'}(a, b; X')$, where p' is conjugate exponent of p, with $p' = \infty$ if $p = 1$, X' is the dual space of X. The duality between $L^p(a, b; X)$ and $L^{p'}(a, b; X')$ becomes

$$\langle u, v\rangle_{L^{p'}(a,b;X')\times L^p(a,b;X)} = \int_a^b \langle u(t), v(t)\rangle_{X'\times X} dt$$

for $u \in L^{p'}(a, b; X')$ and $v \in L^p(a, b; X)$. If $1 \leq p < \infty$, $L^p(a, b; X)$ is a separable space, provided X is a separable space. For $1 < p < \infty$, $L^p(a, b; X)$ is reflexive if so is X. The spaces $L^p(a, b; L^p(\Omega))$ and $L^p((a, b) \times \Omega)$ are isometrically isomorphic and, in this sense $L^p(a, b; L^p(\Omega)) = L^p((a, b) \times \Omega)$.

Likewise, we denote by $C^k(a, b; X)$ the class of all functions from (a, b) to the Banach space X, which are differentiable in (a, b) up to the integer non-negative order k.

Let Ω be a domain in $\mathbb{R}^N$, $1 \leq p \leq \infty$ and k an integer non-negative. The spaces $\mathbf{C}^k(\Omega)$, $\mathbf{C}_0^\infty(\Omega)$, $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$, $\mathbf{W}_0^{k,p}(\Omega)$, $\mathbf{H}^k(\Omega)$, $\mathbf{H}_0^k(\Omega)$, $\mathbf{W}^{1-1/p,p}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$, *etc.*, consist of those functions $\mathbf{u} : \Omega \to \mathbb{R}^M$, $\mathbf{u} = (u_1, \dots, u_M)$, with u_i in $C^k(\Omega)$, $C_0^\infty(\Omega)$, $L^p(\Omega)$, $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$, $H^k(\Omega)$, $H_0^k(\Omega)$, $W^{1-1/p,p}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, *etc.*, respectively, for all $i = 1, \dots, N$.

Bibliography

[1] R.A. Adams. *Sobolev Spaces*, volume 65 of *Pure and Applied Mathematics*. Academic Press, Inc., New York, 2nd edition, 1975.

[2] K.A. Ames, L.E. Payne, and P.W. Schaefer. Spatial decay estimates in time-dependent Stokes flow. *SIAM J. Math. Anal.*, 24(6):1395–1413, 1993.

[3] C.J. Amick and L.E. Fraenkel. Steady solutions of the Navier-Stokes equations representing plane flow in channels of various types. *Acta Math.*, 144(1-2):83–152, 1980.

[4] C. Amrouch, V. Girault, M.E. Schonbeck, and T.P Schonbeck. Pointwise decay of solutions and of higher derivatives to Navier-Stokes equations. *SIAM J. Math. Anal.*, 31(4):740–753, 2000.

[5] S.N. Antontsev. On the localization of solutions of nonlinear degenerate elliptic and parabolic equations. *Soviet Math. Dokl.*, 24(2):420–424, 1981.

[6] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative external field: I. The stationary Stokes equations. In *Book of Abstracts NSEC8*, pages 55–56, St.-Petersburg, Russia, 11-18 Sep., 2002. EIMI.

[7] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. On the confinement of a viscous fluid by means of a feedback external field. *C. R. Mecanique*, 330:797–802, 2002.

[8] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: I. The stationary Stokes problem. *J. Math. Fluid Mech.*, To appear.

[9] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: II. The stationary Navier-Stokes problem. *Rend. Lincei Mat. Appl.*, To appear.

[10] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: thermal effects without phase changing. In *Proceedings TPDE*, Óbidos, Portugal, Jun. 7-10, 2003 (To appear).

[11] S.N. Antontsev, J.I. Díaz, and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: III. The evolutionary Navier-Stokes problem. (In preparation).

[12] S.N. Antontsev, J.I. Díaz, and S. Shmarev. *Energy Methods for Free Boundary Problems*, volume 48 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser, Boston, 2002.

[13] S.N. Antontsev and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: Navier-Stokes problem. In *Book of Abstracts PTFBP*, pages 3–4, Kiev, Ukraine, 19-27 Aug., 2003. IM-NASU.

[14] S.N. Antontsev and H.B. de Oliveira. Stopping a viscous fluid by a feedback dissipative field: stationary Navier-Stokes problem. In *Book of Abstracts NPDE*, page 13, Alushta-Donetsk, Ukraine, 15-21 Sep., 2003. IAMM-NASU.

[15] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov. *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*. North-Holland, Amsterdam, 1990.

[16] G.K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, reprint of 1967 edition, 1994.

[17] Ph. Benilan, H. Brézis, and M.G. Crandall. A semilinear equation in $L^1(\mathbb{R}^n)$. *Ann. Scuola Norm. Sup. Pisa*, 2:523–555, 1975.

[18] F. Bernis. Compactness of the support for some nonlinear elliptic problems of arbitrary order in dimension n. *Comm. Partial Differential Equations*, 9(3):271–312, 1984.

[19] F. Bernis. Extinction of the solutions of some quasilinear elliptic problems of arbitrary order. *Proc. Symp. Pure Math.*, 45:125–132, 1986. Part 1.

[20] F. Bernis. Finite speed of propagation and asymptotic rates for some nonlinear higher order parabolic equations with absorption. *Proc. Royal Soc. Edinburgh Sect. A*, 104:1–19, 1986.

[21] F. Bernis. Qualitative properties for some nonlinear higher order degenerate parabolic equations. *Houston J. Math.*, 14(3):319–352, 1988.

[22] F. Bernis. Elliptic and parabolic semilinear problems without conditions at infinity. *Arch. Ration. Mech. Anal.*, 106:217–241, 1989.

[23] W. Borchers and T. Miyakawa. L^2-decay for Navier-Stokes equations in unbounded domains, with applications to exterior stationary domains. *Arch. Ration. Mech. Anal.*, 118:273–295, 1992.

[24] L. Brandolese. Localization de la vorticité et applications au comportement asymptotique des solutions de Navier-Stokes. *Journées Équations aux Dérivées Partielles, Univ. Nantes*, Exp. No. III:13 pp., 2002.

[25] J. Brannan, J. Duan, and T. Wanner. Dissipative quasigeostrophic dynamics under random forcing. *J. Math. Anal. Appl.*, 228(1):221–233, 1998.

[26] H. Brézis. Solutions with compact support of variational inequalities. *Uspehi Mat. Nauk*, 129:103–108, 1974.

[27] H. Brézis and F.E. Browder. Strongly nonlinear elliptic boundary value problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5:587–603, 1978.

[28] H. Brézis and F.E. Browder. Some properties of higher order Sobolev spaces. *J. Math. Pures Appl.*, 61(3):245–259, 1982.

[29] J.R. Canon, E. DiBenedetto, and G.H. Knightly. The steady-state Stefan problem with convection. *Arch. Ration. Mech. Anal.*, 73:79–97, 1980.

[30] J.R. Canon, E. DiBenedetto, and G.H. Knightly. The bidimensional Stefan problem with convection: the time dependent case. *Comm. Partial Differential Equations*, 8:1549–1604, 1983.

[31] J. Carrillo and M. Chipot. On some nonlinear elliptic equations involving derivatives of the nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A*, 100(3-4):281–294, 1985.

[32] A.-L. Cauchy. Sur l'equilibre et le mouvement d'un système de points materials sollicités par forces d'actraction ou de pépulsion mutuelle. In *Exercices des Mathematics*, volume 3, pages 187–213. Freres DeBure, Paris, 1826-30.

[33] P. Constantin and C. Foias. *Navier-Stokes Equations*. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago, 1988.

[34] J.I. Díaz. *Nonlinear Partial Differential Equations and Free Boundaries: Elliptic Equations*, volume 106 of *Research Notes in Mathematics*. Pitman, London, 1985.

[35] J.I. Díaz. Un problema de frontera libre en el estudio del equilibrio magnetohidrodinamico de un plasma en una configuracion Stellarator. In J.I. Díaz and A. Galindo, editors, *Modelos Matematicos en Física de Plasmas*, volume XXX of *Serie de Ciencias Exactas*, pages 73–132, Madrid, 1995. RACEFNM, CIEMAT.

[36] J.I. Díaz. Energy methods for free boundary problems: new results and some remarks on numerical algorithms. To appear in ESAIM Proc., 2003.

[37] J.I. Díaz and M.A. Herrero. Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems. *Proc. Royal Soc. Edinburgh Sect. A*, 89:249–258, 1981.

[38] J.I. Díaz and L. Véron. Compacité du support des solutions d'équations quasi linéaires elliptiques ou paraboliques. *C. R. Acad. Sci. Paris Sér. I Math.*, 297:149–152, 1983.

[39] J.I. Díaz and L. Véron. Local vanishing properties of solutions of elliptic and parabolic equations. *Trans. Amer. Math. Soc.*, 290(2):787–814, 1985.

[40] N. Dunford and J.T. Shwartz. *Linear operators. Part I. General theory*, volume 75 of *Wiley Classics Library*. John Wiley & Sons, Inc., New York, reprint of the 1958 original edition, 1988.

[41] A.R. Elcrat and V.G. Sigillito. A spatial decay estimate for the Navier-Stokes equations. *Z. Angew. Math. Phys.*, 30(3):449–455, 1979.

[42] L.È. El'sgol'c. *Qualitative Methods in Mathematical Analysis*, volume 126 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, 1964.

[43] L. Euler. Principes géneraux du mouvement des fluides. *Mém. Acad. Sci. Berlin*, 11:274–315, 1755.

[44] L.C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, 1998.

[45] M. Feistauer. *Mathematical Methods in Fluid Dynamics*, volume 67 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1993.

[46] J.P. Freidberg. Ideal magnetohydrodinamic theory of magnetic fusion systems. *Rev. Mod. Phys.*, 54(3), July 1982.

[47] K.O. Friedrichs. *Kurt Otto Friedrichs: Selecta*. Contemporary Mathematicians. Birkhäuser, Boston, 1986.

[48] E. Gagliardo. Ulteriori proprietà di alcune classi funzioni in più variabili. *Ricerche Mat.*, 8:24–51, 1959.

[49] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearized Steady Problems*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.

[50] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Nonlinear Steady Problems*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.

[51] G.P. Galdi. An introduction to the Navier-Stokes initial-boundary value problem. In G.P. Galdi, J.H. Heywood, and R. Rannacher, editors, *Fundamental Directions in Mathematical Fluid Mechanics*, pages 1–70, Basel, 2000. Birkhäuser Verlag.

[52] G.P. Galdi and S. Rionero. *Weighted Energy Methods in Fluid Dynamics and Elasticity*. Number 1134 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1985.

[53] T. Gallay and C.E. Wayne. Long-time asymptotics of the Navier-Stokes and vorticity equations in $\mathbb{R}^3$. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 360(1799):2155–2188, 2002.

[54] H. Gao and J. Duan. Dynamics of quasi-geostrophic fluid motion with rapidly oscillating Coriolis force. *Nonlinear Anal. Real World Appl.*, 4(1):127–138, 2003.

[55] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *A Series of Comprehensive Studies in Mathematics*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 1998.

[56] A.E. Gill. *Atmosphere-Ocean Dynamics*, volume 30 of *International Geophysics Series*. Academic Press, Inc., San Diego, California, 1982.

[57] E.P. Gorbunov and K.A. Razumova. Effect of a strong magnetic field on the magnetohydrodynamic stability of a plasma and the confinement of charged particles in the 'Tokamak' machine. *J. Nucl. Energy, Part C Plasma Phys.*, 6(5):515–525, 1964.

[58] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, Boston, 1985.

[59] J. Hadamard. *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, New Haven, Connecticut, 1923.

[60] G. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, Cambridge, reprint of the 1952 edition, 1997.

[61] E. Hopf. Über die Anfangswertaufabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[62] C.O. Horgan. Plane entry flows and energy estimates for the Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 68:359–381, 1978.

[63] D.D. Joseph. *Stability of Fluid Motions I*, volume 27 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, Berlin Heidelberg, 1976.

[64] D.D. Joseph. *Stability of Fluid Motions II*, volume 28 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, Berlin Heidelberg, 1976.

[65] J.W. Kane and M.M. Sternheim. *Physics*. John Wiley & Sons, New York, 3rd edition, 1988.

[66] T. Kato. Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Math. Z.*, 187(4):471–480, 1984.

[67] G.H. Knightly. On a class of global solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 21:211–245, 1966.

[68] J.K. Knowles. On Saint-Venant's Principle in the two-dimensional linear theory of elasticity. *Arch. Ration. Mech. Anal.*, 21:1–22, 1966.

[69] H. Kozono and T. Ogawa. Decay properties of strong solutions for the Navier-Stokes equations in two-dimensional unbounded domains. *Arch. Ration. Mech. Anal.*, 122:1–17, 1993.

[70] A. Kufner, O. John, and S. Fučik. *Function Spaces*, volume 3 of *Monographs and Textbooks on Mechanics of Solids and Fluids*. Noordhoff-Academia, Leyden-Prague, 1977.

[71] P. K. Kundu. *Fluid Mechanics*. Academic Press, San Diego, California, 1990.

[72] O.A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Fluids*, volume 2 of *Mathematics and its Applications*. Gordon and Breach Science Publishers Inc., New York, revised 2nd english edition, 1969.

[73] O.A. Ladyzhenskaya. The mathematical theory of viscous incompressible fluids in XX century. Seminar given at Center of Mathematics and Fundamental Applications, University of Lisbon, November 07, 2001.

[74] O.A. Ladyzhenskaya and N.N. Ural'tseva. *Linear and Quasilinear Elliptic Equations*, volume 46 of *Mathematics in Science and Engineering*. Academic Press, Oxford, 1968. Translated from the Russian by Scripta Technica.

[75] L.D. Landau, E.M. Lifschitz, and L.P. Pitaevskiĭ. *Electrodynamics of Continuous Media*, volume 8 of *Landau and Lifschitz Course of Theoretical Physics*. Pergamon Press, Oxford, 2nd edition, 1984.

[76] J. Leray. *Jean Leray Oeuvres Scientifiques*. Springer-Verlag, Berlin, 1997. Tome II: Équations aux dérivées partielles réelles et mécanique des fluides.

[77] A.E. Lifschitz. *Magnetohydrodynamics and Spectral Theory*, volume 4 of *Developments in Electromagnetic Theory and Applications*. Kluwer Academic Publishers, Dordrecht, 1989.

[78] C.H. Lin. Spacial decay estimates and energy bounds for the Stokes flow equation. *SAACM*, 2(3):249–264, 1992.

[79] J.-L. Lions. *Quelques Méthodes de resolution des problémes aux limites non linеaires*. Dunod, Paris, 1969.

[80] J.-L. Lions and G. Prodi. Un théorème d'existence et unicité dans les équations de navier- stokes en dimension 2. *C. R. Acad. Sci. Paris*, 248:3519–3521, 1951.

[81] J.-L. Lions, R. Temam, and S. Wang. On mathematical problems for the primitive equations of the ocean: the mesoscale midlatitude case. *Nonlinear Anal. Ser. A: Theory Methods*, 40(1-8):439–482, 2000.

[82] P.-L. Lions. *Mathematical Topics in Fluid Mechanics*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. Clarendon Press, Oxford, 1996. Volume 1 - Incompressible Models.

[83] A.M. Lyapunov. *The General Problem of the Stability of Motion*. Translated and edited by A.T. Fuller. Taylor & Francis, London, 1992.

[84] M. Marion and R. Temam. Navier-stokes equations: Theory and approximation. In P.G. Ciarlet and J.-L. Lions, editors, *Handbook of Numerical Analysis*. Elsevier, North-Holland, Amsterdam, 1998. Numerical Methods for Fluids (Part 1).

[85] V.G. Maz'ja. *Sobolev Spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin Heidelberg, 1985.

[86] J.M. Mihaljan. A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophys. J.*, 136:1126–1133, 1962.

[87] J. Mossino. On some nonlocal models for the equilibrium of a confined plasma. In J.I. Díaz and A. Galindo, editors, *Modelos Matematicos en Física de Plasmas*, volume XXX of *Serie de Ciencias Exactas*, pages 145–156, Madrid, 1995. RACEFNM, CIEMAT.

[88] R. Motley, C.D. Lustig, and S. Sanders. Synchrotron radiation from runaway electrons in the Stellarator. *J. Nucl. Energy, Part C Plasma Phys.*, 3(1):17–21, 1961.

[89] C. Navier. Mémoir sur les lois du mouvement des fluids. *Mem. Acad. Sci. Inst. de France*, 6(2), 1822.

[90] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques*. Masson-Academia, Paris-Prague, 1967.

[91] L. Nirenberg. On elliptical partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 13:115–162, 1959.

[92] O.A. Oleinik, A.S. Kalashnikov, and Chzhou Yui-Lin. The Cauchy problem and boundary problems for equations of the type of non-stationary filtration. *Izv. Akad. Nauk SSSR Ser. Mat.*, 22:667–704, 1958.

[93] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer-Verlag, New York, 2nd edition, 1987.

[94] S.-D. Poisson. Mémoire sur les équations génêrales de l'équilibre et du movement des corps solides élastiques et des fluids. *Journal de l'École Polytéchnique*, 13:1–174, 1831.

[95] K.R. Popper. *The Open Universe: An Argument for Indeterminism*. Rowman and Littlefield, 1985.

[96] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin Heidelberg, 1994.

[97] M. Růžička. *Electrorheological Fluids: Modelling and Mathematical Theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin Heidelberg, 2000.

[98] J.-F. Rodrigues. A steady-state Boussinesq-Stefan problem with continuous extraction. *Ann. Mat. Pura Appl.*, 144:203–218, 1986.

[99] J.-C. Saint-Venant. Note à joindre au mémoire sur la dynamique des fluides. *Comptes Rendus Hebdomadaires des Seances de l'Acadamie des Sciences*, 17:1240–1243, 1843.

[100] S. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integral Transforms Spec. Funct.* To appear.

[101] M.E. Schonbeck. Large time behavior of solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 11(7):733–763, 1986.

[102] M.E. Schonbeck. Lower bounds of rates of decay for solutions to the Navier-Stokes equations. *J. Amer. Math. Soc.*, 4(3):423–449, 1991.

[103] A.P.S. Selvadurai. *Partial Differential Equations in Mechanics 2: The Biharmonic Equation, Poisson's Equation.* Springer-Verlag, Berlin Heidelberg, 2000.

[104] J. Serrin. Mathematical principles of classical fluid mechanics. *Handbuch der Physik - Encyclopedia of Physics*, VIII(1):125–263, 1959.

[105] J. Serrin. The solvability of boundary value problems. *Proc. Symp. Pure Math.*, 28:507–524, 1976.

[106] V. Sheffer. A solution to the Navier-Stokes equations inequality with an internal singularity. *Commun. Math. Phys.*, 101:47–85, 1985.

[107] A.E. Shishkov. Propagation of perturbations in the singular Cauchy problem for quasilinear degenerate parabolic equations. *Sb. Math.*, 187(9):1391–1410, 1996.

[108] A.E. Shishkov. Dead cores and instantaneous compactification of supports of energy solutions of quasilinear parabolic equations of arbitrary order. *Sb. Math.*, 190(11-12):1843–1869, 1999.

[109] W.R. Showalter. *Mechanics of Non-Newtonian Fluids.* Pergamon Press, Oxford, 1978.

[110] H. Sohr. *The Navier-Stokes Equations, An Elementary Functional Analytic Approach.* Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel, 2001.

[111] G. Stokes. On the theories of the internal friction of fluids in motion. *Trans. Cambridge Phil. Soc.*, 9, 1845.

[112] B. Straughan. *The Energy Method, Stability and Nonlinear Convection*, volume 91 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1992.

[113] S. Takahashi. A weighted equation approach to decay rate estimates for the navier-stokes equations. *Nonlinear Anal. Ser. A: Theory Methods*, 37(6):751–789, 1999.

[114] J.B. Taylor. Rotation and instability of plasma in fast B_ζ compression experiments. *J. Nucl. Energy, Part C Plasma Phys.*, 4:401–407, 1962.

[115] R. Temam. *Navier-Stokes Equations Theory and Numerical Analysis*, volume 2 of *Studies in Mathematics and its Applications*. North-Holand, Amsterdam, revised edition, 1979.

[116] R. Temam. Some developments on Navier-Stokes equations in the second half of the 20th century. In *Development of Mathematics 1950-2000*, pages 1049–1106. Birkhäuser, Basel, 2000.

[117] R.A. Toupin. Saint-Venant's Principle. *Arch. Ration. Mech. Anal.*, 18:83–96, 1965.

[118] C. Truesdell. Notes on the history of the general equations of hydrodynamics. *Am. Math. Monthly*, 60:445–458, 1953.

[119] C. Truesdell and W. Noll. The non-linear field theories of mechanics. *Handbuch der Physik - Encyclopedia of Physics*, III(3), 1965.

[120] I.I. Vrabie. *Compactness Methods for Nonlinear Evolutions*, volume 75 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 2n edition, 1995.

[121] W. Walter. Ganze Lösungen der Differentialgleichung $\triangle^p u = f(u)$. *Math. Z.*, 67:32–37, 1957.

[122] M. Wiegner. Decay results for weak solutions of the Navier-Stokes equations on $\mathbb{R}^n$. *Comm. Partial Differential Equations*, 35(2):303–313, 1987.