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MATHEMATICAL MODELS IN DYNAMICS OF NON-NEWTONIAN FLUIDS AND IN GLACIOLOGY

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Abstract. *This paper deals with the study of some qualitative properties of solutions of mathematical models in non-Newtonian isothermal fluid flows and in theoretical glaciology. In the first type of models, we consider the extinction in a finite time of the solutions by using a global energy method. We prove that this property holds for pseudo-plastic fluids or for the general class of Newtonian and dilatant fluids, assumed the presence of a dissipation term (which may have an anisotropic nature and can vanish in, at most, one spatial direction). In the case of the ice sheet model in Glaciology (with a formulation involving a quasi-linear degenerate equation similar to the ones arising in non-Newtonian flows), we analyze the behavior of the free boundary (given by the support of the height h of the ice sheet) for different cases and according to the values of the ablation function and the initial hight. We use here some other energy methods of a local nature and so completely different to the method used in the first part of the paper.*

Part I
Localization effects in a general fluid dynamics model

1 Introduction

From the basic principles of Fluid Mechanics, it is well known that, in isothermal motions of incompressible fluids with no inner mass sources, the velocity field and pressure are determined from:

- the incompressibility condition
$$\mathrm{div}\mathbf{u} = 0; \tag{1.1}$$
- the conservation of mass
$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{u}) = 0; \tag{1.2}$$
- the conservation of momentum
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \rho\,\mathbf{f} + \mathrm{div}\mathbf{S}. \tag{1.3}$$

In this part, we consider the mathematical problem posed by (1.1)-(1.3) in a cylinder

$$Q_T := \Omega\times(0,T)\subset\mathbb{R}^N\times\mathbb{R}^+\,,$$

where Ω is a bounded domain whose boundary $\partial\Omega$ is assumed to be smooth enough. The boundary of Q_T is defined by

$$\Gamma_T := (0,T)\times\partial\Omega\,.$$

The dimensions of physical interest are $N=2$ and $N=3$, but the results to be presented here extend to any dimension $N\geq 2$. We consider a general class of non-Newtonian fluid problems for which the stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = -p\mathbf{I} + \mathbf{F}(\mathbf{D}),\quad \mathbf{D} = \frac{1}{2}\left(\nabla\,\mathbf{u} + \nabla\,\mathbf{u}^T\right)\,, \tag{1.4}$$

where $\mathbf{F}$ is a symmetric tensor and which is assumed to satisfy, for all $\mathbf{u}\in\mathbb{R}^N$,

$$\delta\;|\mathbf{D}(\mathbf{u})|^q \leq \mathbf{D}(\mathbf{u}):\mathbf{F}(\mathbf{u}) \equiv \sum_{i,j=1}^{N} F_{ij}D_{ij},\quad 0<\delta=\delta(\rho)<\infty,\ 1\leq q<\infty\,. \tag{1.5}$$

Fluids satisfying (1.4)-(1.5) are called viscous-plastic if $1\leq q<2$ and dilatant if $q>2$. Other names found in the literature are pseudo-plastic for $1<q<2$ and Bingham fluids when $q = 1$. If $q = 2$, the fluid is Newtonian and we fall in the classical Navier-Stokes equations. In this case, for incompressible homogeneous viscous fluids the stress tensor $\mathbf{S}$ has the form

$$\mathbf{S} = -p\mathbf{I} + 2\mu\mathbf{D}\,,$$

where μ is the dynamical viscosity. The notation used in (1.1)-(1.4) is well known: $\mathbf{u}$ is the velocity field, p is the pressure, ρ is the density, $\mathbf{f}$ is the forcing term, $\mathbf{D}$ is the tensor of rate of deformations and $\mathbf{I}$ is the unit tensor. System (1.1)-(1.4) is endowed with the initial and boundary conditions:

$$\mathbf{u} = \mathbf{u}_0, \qquad \rho = \rho_0 \qquad \text{in} \quad \Omega \quad \text{when } t = 0; \tag{1.6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_T. \tag{1.7}$$

In this work, we consider a forces field $\mathbf{f}$ in (1.3) such that

$$\mathbf{f}(\mathbf{x}, t, \mathbf{u}) = \mathbf{h}(\mathbf{x}, t, \mathbf{u}) + \mathbf{g}(\mathbf{x}, t), \tag{1.8}$$

where $\mathbf{g}$ is a given function and $\mathbf{h}$ depends nonlinearly on the velocity field $\mathbf{u}$. We consider two different possibilities for the function $\mathbf{h}$: the case of isotropic dissipation

$$-\mathbf{h}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{u} \geq c|\mathbf{u}|^{\sigma} \quad \forall\ \mathbf{u} \in \mathbb{R}^N, \qquad \sigma \in (1, 2)\,, \tag{1.9}$$

for some positive constant c; and the case of anisotropic dissipation

$$-\mathbf{h}(\mathbf{x}, t, \mathbf{u}) \cdot \mathbf{u} \geq \sum_{i=1}^{N} c_i\, |u_i|^{\sigma_i} \quad \forall\ \mathbf{u} \in \mathbb{R}^N, \qquad \sigma_i \in (1, 2), \tag{1.10}$$

for some non-negative constants c_i, with $i = 1, \ldots, N$. In both cases the function $\mathbf{g}$, considered in (1.8), satisfies to

$$\|\mathbf{g}(\cdot, t)\|_{2,\Omega} \leq C\,(1 - t/t_{\mathbf{g}})_+^{\nu}\ , \tag{1.11}$$

for some positive constants C, ν, $t_{\mathbf{g}}$. Relation (1.11) means the forces field $\mathbf{g}$ vanishes at the instant of time $t = t_{\mathbf{g}}$. From the Fluid Mechanics point of view, conditions (1.9) and (1.10) mean the forces field $\mathbf{f}$ is a feedback term. This feedback is presented as an isotropic condition in (1.9) and as anisotropic in (1.10). In (1.9) the dissipation of $\mathbf{f}$ does not vary from one direction to another. But, in (1.10) the dissipation may be different for distinct directions. Moreover, in the former case and from condition (1.10), we can say the feedback forces field $\mathbf{h}$, and thus $\mathbf{f}$, is dissipative, in order to each component u_k, in all directions x_k where $c_k > 0$, for $k = 1, \ldots, N$. In the simpler case of $\mathbf{g} \equiv \mathbf{0}$, we have the following examples of forces fields $\mathbf{f}$ satisfying (1.9) or (1.10):

$$\mathbf{f}(\mathbf{x}, t, \mathbf{u}) = -c|\mathbf{u}|^{\sigma-2}\mathbf{u};$$

$$\mathbf{f}(\mathbf{x},t,\mathbf{u}) = -(c_1\,|u_1|^{\sigma_1-2}u_1,\ldots,c_N\,|u_N|^{\sigma_N-2}u_N)\,.$$

The motivation for the consideration of a forces field satisfying to (1.9), or (1.10), is purely mathematical and goes back to the works of Benilan *et al.* [11], Díaz and Herrero [17], and Bernis [12, 13]. These authors studied the importance of the absorption term $|u|^{\sigma-2}u$ in order to prove qualitative properties related with compact supported solutions, or solutions which exhibit finite speed of propagations, or which extinct in time, for different initial-boundary value problems. Similar problems to (1.1)-(1.8) were considered by the authors in a variety of problems in the scope of Fluid Mechanics in [1]-[8]. It should be remarked that questions of time behavior of solutions to the homogeneous version of (1.1)-(1.7) have been studied by many authors (see [1] and the references therein). All these references are only concerned with exponential decays and the better we saw are related with power spatial and temporal decays in different norms. In spite of many work in this field, so far, and to the best of our knowledge, there are no results establishing the extinction of solutions to these problems in a finite time.

2 Weak formulation

In this section we define the class of solutions we shall work with and give some remarks in how to prove the existence and uniqueness results. We cannot omit all the mathematics needed to handle these issues, but always we can we will avoid the technical parts, which in turn are the most difficult to understand for someone out of this field. We are interested in a class of solutions $(\rho,\mathbf{u})$ to the problem (1.1)-(1.8) such that

$$E(t)+\int_\Omega |\nabla\mathbf{u}|^q\,d\mathbf{x}<\infty,\quad \text{where}\quad E(t):=\frac{1}{2}\int_\Omega \rho(\mathbf{x},t)\,|\mathbf{u}(\mathbf{x},t)|^2\,d\mathbf{x}, \tag{2.12}$$

$$1/M\leq\rho\leq M,\qquad M=const.>0\,. \tag{2.13}$$

To define the notion of solutions we shall consider, we introduce the following function spaces:

$$\mathbf{J}^r(\Omega)=\left\{\mathbf{u}\in\mathbf{L}^r(\Omega):\int_\Omega \mathbf{u}\cdot\nabla\phi\,d\mathbf{x}=0,\quad \nabla\phi\in\mathbf{L}^{r'}(\Omega)\right\}\,;$$

$$\mathbf{W}_{q,\sigma}=\{\mathbf{u}\in\mathrm{L}^\infty\left(0,T;\mathbf{L}^2(\Omega)\right)\cap\mathrm{L}^2(0,T;\mathbf{W}^{1,q}_0(\Omega)):\sigma\in(1,2)\,,\ \mathbf{u}\in\mathrm{L}^\sigma(0,T;\mathbf{L}^\sigma(\Omega))\}\,;$$

$$\begin{aligned}\mathbf{W}_{q,\overline{\sigma}}=&\{\mathbf{u}\in\mathrm{L}^\infty\left(0,T;\mathbf{L}^2(\Omega)\right)\cap\mathrm{L}^2(0,T;\mathbf{W}^{1,q}_0(\Omega)):\overline{\sigma}=(\sigma_1,...,\sigma_N),\\ &\sigma_i\in(1,2)\quad\text{and}\quad u_i\in\mathrm{L}^{\sigma_i}(0,T;\mathrm{L}^{\sigma_i}(\Omega))\quad\forall\ i=1,...,N\}.\end{aligned}$$

For a detailed comprehension of these functions spaces, we address the reader to [8] and the references therein to the monographs by Adams and Maz'ya.

Definition 2.1 *A pair of functions* $(\mathbf{u},\rho)$ *is a weak solution of problem (1.1)-(1.8), if:*
1. $\mathbf{u}\in\mathbf{W}_{q,\sigma}$ *in case (1.9) is fulfilled, or* $\mathbf{u}\in\mathbf{W}_{q,\overline{\sigma}}$ *in case of (1.10), and for every*

$\mathbf{\Phi} \in \mathrm{C}^1\left([0,T];\mathbf{J}^1(\Omega)\right)$ *such that* $\mathbf{\Phi}(\mathbf{x},T)=0$

$$-\int_0^T\int_\Omega \rho\left[\mathbf{u}\cdot\mathbf{\Phi}_t+\mathbf{u}\otimes\mathbf{u}:\nabla\mathbf{\Phi}\right]d\mathbf{x}\,dt+$$
$$\int_0^T\int_\Omega \mathbf{F}(\mathbf{u}):\nabla\mathbf{\Phi}\,d\mathbf{x}\,dt=\int_0^T\int_\Omega \rho\,\mathbf{f}\cdot\mathbf{\Phi}\,d\mathbf{x}\,dt-\int_\Omega \rho_0\,\mathbf{u}_0\cdot\mathbf{\Phi}(0)\,d\mathbf{x}\,;$$

2. ρ *satisfies (2.13) and for every* $\varphi \in \mathrm{C}^1\left([0,T];\mathrm{H}^1(\Omega)\right)$ *such that* $\varphi(\mathbf{x},T)=0$

$$\int_0^T\int_\Omega \rho\left[\varphi_t+(\mathbf{u}\cdot\nabla)\varphi\right]d\mathbf{x}\,dt+\int_\Omega \rho_0\varphi(0)\,d\mathbf{x}=0\,.$$

According to [8, §4.7] and references therein, problem (1.1)-(1.8) has, at least, a weak solution, if the mass force term does not depend on $\mathbf{u}$, *i.e* if we consider $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$ in a suitable function space. Moreover, at least formally, every weak solution of problem (1.1)-(1.8) satisfies the energy relation

$$\frac{d}{dt}E(t)+\int_\Omega \mathbf{F}(\mathbf{u}):\mathbf{D}(\mathbf{u})\,d\mathbf{x}=\int_\Omega \rho\,\mathbf{f}\cdot\mathbf{u}\,d\mathbf{x}, \tag{2.14}$$

where $E(t)$ is given in (2.12). The formal derivation of (2.14) relies on (1.1)-(1.4), the symmetry of the tensor $\mathbf{F}$, integration-by-parts formulae and boundary condition (1.7). By the existence theory of a weak solution to problem (1.1)-(1.7), is known that every weak solution satisfies to the energy equality (2.14) only if $N=2$. For $N\geq 3$ this is no longer valid. But, for our purposes in this paper, it is enough that (2.14) is verified with the equality sign = replaced by the inequality one $\leq$. And this is satisfied for every such weak solution and for every dimension $N\geq 2$. In our further study we assume the existence of, at least, a weak solution of problem (1.1)-(1.8) in the sense of Definition 2.1. We give here only the ideas of the proof. We consider the four main cases: $q=1$, $1<q<2$, $q=2$ and $q>2$. For each one of such cases, and for different constitutive laws, there are known existence results for suitable forces given in appropriated function spaces. In [14] is proved the existence of a weak solution for $q=1$ and $\mathbf{S}=-p\mathbf{I}+\gamma\mathbf{D}+gD_{II}^{-1/2}\mathbf{D}$. For $1\leq q<2$ and $\mathbf{S}=-p\mathbf{I}+2\mu\mathbf{D}+\alpha D_{II}^{(q-2)/2}\mathbf{D}$, the existence of a weak solution is proved in [18]. In [9] was proved the existence of a weak solution for $\mathbf{S}=-p\mathbf{I}+2\mu\mathbf{D}$ (and $q=2$). Finally, for $\mathbf{S}=-p\mathbf{I}+2\mu\mathbf{D}+\alpha D_{II}^{(q-2)/2}\mathbf{D}$ and $q\geq 2$, the existence of a weak solution was proved in [18]. In our problem (1.1)-(1.8), the idea is to use energy relation (2.14) (see also (3.26) below), assumption (2.13), repeating the correspondingly arguments of [9, 14, 18, 19] (see also [26]) and to use a fixed point argument. Simpler proofs can be shown if, instead of (1.9), or (1.10), the forcing field (1.8) is given, respectively, by

$$\mathbf{f}(\mathbf{x},t,\mathbf{u})=-c|\mathbf{u}|^{\sigma-2}\mathbf{u}+\mathbf{g}(\mathbf{x},t)\,,\quad \sigma\in(1,2)\,,\quad c=const.>0\,, \tag{2.15}$$

or

$$\begin{aligned} &\mathbf{f}(\mathbf{x},t,\mathbf{u}) = -(c_1\,|u_1|^{\sigma_1-2}u_1,\ldots,c_N\,|u_N|^{\sigma_N-2}u_N) + \mathbf{g}(\mathbf{x},t)\,, \\ &\sigma_i \in (1,2)\,, \quad c_i = const. \geq 0 \quad (i=1,\ldots,N)\,, \end{aligned} \tag{2.16}$$

where $\mathbf{g}$ satisfies to (1.11). In these cases, assuming that $\mathbf{g}$ is given in a suitable function space, we can adapt the correspondingly proofs written in [9, 14, 18, 19]. The only difference lies in the first term of (2.15), or (2.16). We only need to prove that the corresponding Galerkin approximations converge. If $N=2$, we can also adapt the correspondingly results of [9, 14, 18, 19] to prove the uniqueness of solutions. In this case, we only need to apply properly the following result to show the monotonicity of the resulting integral terms of (2.15) or (2.16).

Lemma 2.1 *For all $p\in(1,\infty)$ and $\delta\geq 0$, there exist constants C_1 and C_2, depending on p and N, such that for all $\xi,\ \eta\in\mathbf{R}^N$, $N\geq 1$,*

$$\left||\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right| \leq C_1|\xi-\eta|^{1-\delta}\,(|\xi|+|\eta|)^{p-2+\delta} \tag{2.17}$$

and

$$\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta) \geq C_2|\xi-\eta|^{2+\delta}\,(|\xi|+|\eta|)^{p-2-\delta} \tag{2.18}$$

See [10] for the proof and also the references cited therein for other forms of (2.17) and (2.18).

3 Extinction in a finite time

In this section we shall study the time asymptotic behavior of solutions to (1.1)-(1.8), when one considers a forces field satisfying to (1.9) or (1.10). The time property we are going to establish is usually denominated as the extinction in a finite time and can be defined as follows.

Definition 3.1 *We say the weak solutions $(\mathbf{u},\rho)$ of the problem (1.1)-(1.8) possesses the extinction in a finite time property if there exists (a finite time) $t^*\in(0,\infty)$ such that $\mathbf{u}(\mathbf{x},t)=\mathbf{0}$ a.e. in Ω and for all $t\geq t^*$.*

To establish this property, we shall make use of two important results quite used in Continuum Mechanics or in its mathematical treatment.

Lemma 3.1 *Let Ω be a domain of $\mathbb{R}^N$, $N\geq 2$, with a locally Lipschitz compact boundary $\partial\Omega$, and assume $1\leq q<\infty$. If $\mathbf{u}\in\mathbf{W}^{1,q}_0(\Omega)$, then the following inequality holds*

$$\frac{1}{C}\|\nabla\mathbf{u}\|_{q,\Omega} \leq \|\mathbf{D}(\mathbf{u})\|_{q,\Omega} \leq C\|\nabla\mathbf{u}\|_{q,\Omega}, \quad C=C(q,N)\,, \tag{3.19}$$

where $\mathbf{D}$ is the rate of deformations tensor.

This first result is the so-called second Korn's inequality and it extends for suitable unbounded domains. For the proof, see [8] and the references cited therein to the works by Oleinik and Yosifian.

Lemma 3.2 *Let Ω be a domain of $\mathbb{R}^N$, $N \geq 1$, with a compact boundary $\partial\Omega$. Assume that $u \in W_0^{1,p}(\Omega)$. For every fixed number $r \geq 1$, there exists a constant C depending only on N, p and r such that*

$$\|u\|_{q,\Omega} \leq C\|\nabla u\|^{\theta}_{L^p(\Omega)}\|u\|^{1-\theta}_{L^r(\Omega)}, \tag{3.20}$$

where $\theta \in [0,1]$, $p,\ q \geq 1$, are linked by $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$, and their admissible range is:
(1) If $N = 1$, $q \in [r,\infty]$, $\theta \in \left[0, \frac{p}{p+r(p-1)}\right]$;
(2) If $p < N$, $q \in \left[\frac{Np}{N-p}, r\right]$ if $r \geq \frac{Np}{N-p}$ and $q \in \left[r, \frac{Np}{N-p}\right]$ if $r \leq \frac{Np}{N-p}$;
(3) If $p \geq N > 1$, $q \in [r,\infty)$ and $\theta \in \left[0, \frac{Np}{Np+r(p-N)}\right]$.

This one is denoted by Sobolev interpolation embedding and extends also for suitable unbounded domains. For the proof see [8] and the references cited therein to the works by Gagliardo and Nirenberg.

3.1 Viscous-plastic fluids

We start by considering the case of viscous-plastic fluids, *i.e.* we assume $1 \leq q < 2$ in (1.5). In this case, we will see that, to prove the extinction in a finite time property, there will be no need of assumptions (1.9) or (1.10). We merely assume that $\mathbf{h} \equiv \mathbf{0}$ in (1.8), leading us back to the usual problem, with $\mathbf{f}$ depending only on $(\mathbf{x}, t)$.

Theorem 3.1 *Let $(\mathbf{u}, \rho)$ be a weak solution of problem (1.1)-(1.8) in the sense of Definition 2.1. If, in (1.8), $\mathbf{h} \equiv \mathbf{0}$ and:*
1. $\mathbf{g} \equiv 0$, then there exists $t^* > 0$ such that $E(t) = 0$ for almost all $t \geq t^*$ - in particular, $\mathbf{u} \equiv 0$ in $Q \cap \{t \geq t^*\}$;
2. $\mathbf{g} \not\equiv 0$ satisfies

$$\|\mathbf{g}(\cdot,t)\|_{p,\Omega} \leq \gamma \left(1 - \frac{t}{t_{\mathbf{g}}}\right)_{+}^{\frac{q}{2-q}}, \tag{3.21}$$

then there exists a constant $\gamma_0 > 0$ such that $E(t) = 0$ for almost all $t \geq t_g$, if $\gamma \geq \gamma_0 > 0$ - in particular, $\mathbf{u} \equiv 0$ in $Q \cap \{t \geq t_{\mathbf{g}}\}$, if $\gamma \geq \gamma_0 > 0$.

PROOF. The proof follows as in [8, pp. 236-7] by using Korn's inequality (3.19) in the energy relation (2.14). If $\mathbf{g} \equiv 0$, we get the homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C\,(E(t))^{\frac{q}{2}} \leq 0\,, \tag{3.22}$$

where $C = C(q, N, \delta)$ is a positive constant. Then, integrating (3.22) we prove the first assertion.

If $\mathbf{g}$ satisfies to (3.21), we use Hölder's and Young's with $\varepsilon > 0$ inequalities to obtain the non-homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C_1 \left(E(t)\right)^{\frac{q}{2}} \leq C_2 \left(1 - \frac{t}{t_{\mathbf{g}}}\right)_+^{\frac{q}{2-q}}, \tag{3.23}$$

where $C_1 = C_1(q, M, N, \delta, \varepsilon)$ and $C_2 = C_2(q, \gamma, \varepsilon)$ are positive constants. The analysis of (3.23), which have been considered in [8, p. 232], proves the second assertion.□

Remark 3.1 *From the physical point of view, Theorem 3.1 asserts that, for viscous-plastic fluids, the extinction in a finite time property is determined, only, by the structure of the stress tensor. And this is true wether the fluid is generated by the initial data or is stirred by the forces term (3.21).*

3.2 Newtonian and dilatant fluids

In this section, we consider the case of Newtonian and dilatant fluids, *i.e.* we assume

$$q \geq 2 \quad \text{in} \quad (1.5). \tag{3.24}$$

First of all let us derive an estimate which will be useful wether we assume that (1.8) satisfies (1.9) or (1.10). Using assumption (3.24), Hölder's inequality, the vectorial version of Sobolev interpolation embedding (3.20) with $\theta = 1$ and $r = 1$, and Young's inequality with a suitable $\varepsilon > 0$, we get

$$\left| \int_\Omega \mathbf{u} \cdot \mathbf{g} \, d\mathbf{x} \right| \leq \|\mathbf{g}\|_{p,\Omega} \, \|\mathbf{u}\|_{p',\Omega} \leq C \|\mathbf{g}\|_{p,\Omega} \|\nabla \mathbf{u}\|_{q,\Omega} \leq \varepsilon \|\nabla \mathbf{u}\|_{q,\Omega} + C(\varepsilon) \|\mathbf{g}\|_{p,\Omega}^{\frac{q}{q-1}}, \tag{3.25}$$

where $p = Nq/[N(q-1)+q]$ and $C = C(q, N)$ is a positive constant.

3.2.1 The case of isotropic dissipation

Now let us consider the case of isotropic dissipation, *i.e.* we assume that (1.8) and (1.9) hold. First of all, we can prove that, from the energy relation (2.14), using assumption (2.13), Korn's inequality (3.19) and the estimate (3.25), the following estimate hods

$$\begin{gathered} \sup_{0 \leq t \leq T} E(t) + \int_0^T \int_\Omega \left(|\nabla \mathbf{u}|^q + |\mathbf{u}|^\sigma \right) d\mathbf{x} \, dt \leq \\ C \left(E(0) + \int_0^T \left(\int_\Omega |\mathbf{g}(\mathbf{x}, t)|^p \, d\mathbf{x} \right)^{\frac{q}{p(q-1)}} dt \right), \end{gathered} \tag{3.26}$$

where $C = C(q, N, M, \delta)$ is a positive constant. In consequence, every weak solution to problem (1.1)-(1.9) satisfies to

$$E_{q,\sigma}(t) := \int_\Omega (|\nabla \mathbf{u}|^q + |\mathbf{u}|^\sigma)\, d\mathbf{x} \in \mathrm{L}^1[0,T]. \tag{3.27}$$

Theorem 3.2 *Let $(\mathbf{u}, \rho)$ be a weak solution of problem (1.1)-(1.8) in the sense of Definition 2.1. Assume that conditions (1.9), (2.13) and (3.24) are satisfied.*
1. If $\mathbf{g} \equiv 0$, then there exists $t^* > 0$ such that $E(t) = 0$ for almost all $t \geq t^*$ - in particular, $\mathbf{u} \equiv 0$ in $Q \cap \{t \geq t^*\}$.
2. Let $\mathbf{g} \not\equiv 0$ satisfies

$$\|\mathbf{g}(\cdot,t)\|_{p,\Omega} \leq \gamma \left(1 - \frac{t}{t_{\mathbf{g}}}\right)_+^{\frac{q-1}{q(\mu-1)}}, \qquad p = \frac{Nq}{N(q-1)+q}, \quad \mu > 1,$$

where μ depends on p, q, N and σ. Then there exists a constant $\gamma_0 > 0$ such that $E(t) = 0$ for almost all $t \geq t_g$, if $\gamma \geq \gamma_0 > 0$ - in particular, $\mathbf{u} \equiv 0$ in $Q \cap \{t \geq t_{\mathbf{g}}\}$ if $\gamma \geq \gamma_0 > 0$.

PROOF. The proof uses the same ideas developed in [8, p. 238].
Step 1. In the energy relation (2.14), we use assumptions (2.13) and (3.24), and Korn's inequality (3.19), to obtain

$$\frac{d}{dt}E(t) + CE_{q,\sigma}(t) \leq M \int_\Omega \mathbf{u} \cdot \mathbf{g}\, d\mathbf{x}\,, \tag{3.28}$$

where $E_{q,\sigma}(t)$ is given in (3.27) and $C = C(q, M, N, \delta)$ is a positive constant.
Step 2. We use the vectorial version of the Sobolev interpolation embedding (3.20) with $q = 2$, $p = q$, $r = \sigma$, and Young's inequality, to obtain

$$E(t) \leq \frac{M}{2}\|\mathbf{u}\|_{2,\Omega}^2 \leq C \left(\int_\Omega (|\nabla \mathbf{u}|^q + |\mathbf{u}|^\sigma)\, d\mathbf{x}\right)^\mu, \quad \mu = 1 + \frac{q(2-\sigma)}{q(N+\sigma) - N\sigma}\,, \tag{3.29}$$

where $C = C(q, M, N, \sigma)$ is a positive constant. We notice the assumptions $1 < \sigma < 2$ and $q \geq 2 > \frac{2N}{N+1}$ assure that $\mu > 1$.
Step 3. If $g \equiv 0$, gathering (3.28) and (3.29), we come to the homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + CE(t)^{1/\mu} \leq 0 \qquad \text{for all } t \geq 0\,, \tag{3.30}$$

where $C = C(q, N, \delta)$ is a positive constant. Then, integrating (3.30) we prove the first assertion.

If $g \not\equiv 0$, we use the estimates (3.25) and (3.29), assumption (3.21) and the energy relation (3.28), to achieve the nonhomogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C_1 E(t)^{1/\mu} \leq C_2 \left(1 - \frac{t}{t_{\mathbf{g}}}\right)_+^{\frac{q-1}{q(\mu-1)}} \qquad \text{for all } t \geq 0\,, \tag{3.31}$$

where $C_1 = C_1(q, M, N, \delta, \sigma, \varepsilon)$ and $C_2 = C_2(q, \gamma, \sigma, \varepsilon)$ are positive constants. The analysis of (3.31), which have been considered in [8, p. 232], proves the second assertion.□

3.2.2 The case of anisotropic dissipation

Finally, we consider the case of anisotropic dissipation, *i.e.* and we assume that (1.8) and (1.10) hold. We notice that the estimate (3.25) still holds in this case. Then, proceeding as for (3.26), we prove

$$\sup_{0\leq t\leq T} E(t) + \int_0^T \int_\Omega \left(|\nabla \mathbf{u}|^q + \sum_{i=1}^N c_i |u_i|^{\sigma_i}\right) d\mathbf{x}\, dt \leq$$
$$C\left(E(0) + \int_0^T \left(\int_\Omega |\mathbf{g}(\mathbf{x}, t)|^p \, d\mathbf{x}\right)^{\frac{q}{p(q-1)}} dt\right)\,,$$

where $p = Nq/[N(q-1)+q]$ and $C = C(q, N, M, \delta)$ is a positive constant. In consequence, every weak solution to the problem (1.1)-(1.8), (1.10) satisfies to

$$E^N_{q,\sigma_i}(t) := \int_\Omega \left(|\nabla \mathbf{u}|^q + \sum_{i=1}^N c_i |u_i|^{\sigma_i}\right) d\mathbf{x} \in \mathrm{L}^1[0, T].$$

Here, we let the constants c_i show up in these relations, because from hypothesis (see (1.10)) it may happen that one or more c_i can be zero.

In this case, we shall prove the weak solutions of (1.1)-(1.8) are finite time localized, if (3.24) holds and the forces field $\mathbf{h}$, given in (1.8), satisfies to (1.10) and exhibits dissipation effect except in exactly one direction, *i.e.* we assume that in (1.10) $c_i = 0$ for only one $i \in \{1, \dots, N\}$. For the sake of simplicity, we assume that is $c_N = 0$, *i.e.*

$$\text{(1.10) holds with } c_N = 0 \text{ and } c_j \neq 0 \text{ for all } j \neq N. \tag{3.32}$$

To establish the extinction in a finite time property, we need to make a restriction on the shape of the domain Ω.

Hypothesis A. The domain Ω is convex, at least, in the x_N direction.

From this assumption, we can say that each line parallel to the x_N axis intersects the boundary $\partial\Omega$ only on two points, say, $\mathbf{x}^0_+ \equiv (x^0_1, ..., x^0_{N-1}, x^+_N)$ and $\mathbf{x}^0_- \equiv (x^0_1, ..., x^0_{N-1}, x^-_N)$, with $x^-_N \leq x^+_N$.

Theorem 3.3 *Let $(\mathbf{u},\rho)$ be a weak solution of problem (1.1)-(1.8) in the sense of Definition 2.1. Assume that conditions (1.10), (2.13), (3.24) and (3.32), and* ***Hypothesis A*** *are satisfied. Then the same two conclusions of Theorem 3.2 are obtained.*

PROOF. We briefly describe the main ideas of the proof which follows closely the one presented in [2].
Step 1. Proceeding as we did for (3.28), we obtain

$$\frac{d}{dt}E(t)+CE^{N-1}_{q,\sigma_i}(t)\leq\int_\Omega \mathbf{u}\cdot\mathbf{g}\,d\mathbf{x}\,,\qquad E^{N-1}_{q,\sigma_i}(t):=\int_\Omega\left(|\nabla\mathbf{u}|^q+\sum_{i=1}^{N-1}|u_i|^{\sigma_i}\right)d\mathbf{x}\,,\quad (3.33)$$

where $C=C_1(q,M,N,c_0)$ is a positive constant with $c_0=\min_{i=1,\dots,N-1}c_i>0$.
Step 2. Proceeding as we did for (3.29), we use the Sobolev interpolation embedding (3.20) with $q=2$, $p=q$, $r=\sigma_i$, and Young's inequality, to obtain for any scalar component u_i, with $i=1,\dots,N-1$, of $\mathbf{u}$

$$\|u_i\|^2_{2,\Omega}\leq C\left(\int_\Omega\left(|\nabla u_i|^q+|u_i|^{\sigma_i}\right)d\mathbf{x}\right)^{\mu_i},\qquad \mu_i=1+\frac{q(2-\sigma_i)}{q(N+\sigma_i)-N\sigma_i}\,,\qquad (3.34)$$

where $C=C(q,N,\sigma_i)$. Moreover, we notice that the assumptions $1<\sigma_i<2$ and $q\geq 2>\frac{2N}{N+1}$ assure that $\mu_i>1$ for any $i=1,\dots,N-1$. Assuming, without loss of generality, that $\|\mathbf{u}\|^2_{2,\Omega}\leq 1$, we can rewrite (3.34), and for all $i=1,\dots,N-1$, in the form

$$\|u_i\|^2_{2,\Omega}\leq C\left(E^{N-1}_{q,\sigma_i}(t)\right)^{\mu_{N-1}},\qquad \mu_{N-1}=\min_{1\leq i\leq N-1}\mu_i\,,\qquad (3.35)$$

where $E^{N-1}_{q,\sigma_i}(t)$ is given by (3.33).
Step 3. In this step, we need a little bit more regularity: we assume, with no lost of generality, that $\mathbf{u}\in\mathbf{C}([0,T]:\mathbf{C}^2(\Omega))$ (a standard approximation argument allows to consider the general case).
We formally multiply the continuity equation

$$\operatorname{div}\mathbf{u}=0,\qquad \mathbf{u}=(u_1,\dots,u_N)\in\mathbf{J}^q(\Omega),$$

by u_N and integrate by parts over an hyperplane $\Omega(z)\subseteq\mathbb{R}^{N-1}$. Then, after an integration procedure and using Hypothesis A, boundary conditions (1.7), and Hölder's inequality, we achieve to the estimate

$$\|u_N\|^2_{2,\Omega}\leq C\,\|\nabla u_N\|_{q,\Omega}\sum_{i=1}^{N-1}\|u_i\|_{q',\Omega}\leq C\,\|\nabla u_N\|_{q,\Omega}\sum_{i=1}^{N-1}\|u_i\|_{2,\Omega}\,,$$

where $C=C(q,N,\Omega)$. Now, applying (3.35) and (3.33), we came to the inequality

$$\|u_N\|^2_{2,\Omega}\leq C\left(E^{S_N}_{q,\sigma}(t)\right)^{\mu_N},\qquad \mu_N=\frac{1}{q}+\frac{\mu_{N-1}}{2}>1,\qquad (3.36)$$

where $C = C(\|u_i\|_{2,\Omega}, q, N, \sigma_i, \Omega)$ is a positive constant, $i = 1, \dots, N-1$. Finally, combining (3.35) and (3.36), we obtain

$$E(t) \leq C \left(E_{q,\sigma_i}^{N-1}(t)\right)^{\mu}, \quad \mu = \min_{1 \leq i \leq N} \mu_i, \tag{3.37}$$

where $C = C(\|u_i\|_{2,\Omega}, q, M, N, \sigma_i, \Omega)$ is a positive constant, $i = 1, \dots, N-1$. We notice that, from (3.34)-(3.36), we have in (3.37) $\mu > 1$.
Step 4. The proof now follows as we did in the proof of Theorem 3.2.□

Remark 3.2 *In this section we have shown that, for non-viscous-plastic fluids, the structure of the stress tensor alone is no longer responsible for the extinction in a finite time property. For these fluids we only can obtain this property, if we introduce an absorption term in the equations which govern such flows. And this absorption term may account for some kind of sink inside these flows.*

Remark 3.3 *The results established in this part can be extended to unbounded domains satisfying the correspondingly hypotheses. The proof is almost the same, we only need to use the known Korn's and Sobolev interpolation embedding inequalities for these domains.*

Part II

Localization effects in a ice dynamics model

4 Introduction

Ice sheets are vast and slow-moving edifices of solid ice, which are mainly concentrated in Antarctic and much smaller in Greenland. They flow under their own weight by solid state creep processes such as the creep of dislocation in the crystalline lattice structure of the ice. In this resemble rivers, expect they move more slowly and are consequently much thicker. Ice sheets have thickness of several kilometers and move at velocities of 10-100 meters per year. Despite their slow movement and apparent changelessness, ice sheets exhibits various interesting dynamic phenomena. In polar climate regions the snow accumulates on the uplands, is compressed into ice and flows out to cover the region under the action of gravity. Ice flows as highly viscous solids from the central parts, where the thickness is great, towards the margins. If the margins are near the coast, it can be formed floating ice shelves. The ice sheet equilibrium can be maintained through a balance between accumulation in the center and ablation at the margins. Accumulation occurs mainly through solid precipitation and ablation can occur either through evaporation or melting of the ice in the warmer climate at the margin, or through calving of icebergs. An interesting phenomenon occurs in ice sheets, where one sees drainage of the ice toward the coast occurring through a series of ice streams, which are highly crevassed rapid flows on the order of 50 Km wide, bounded by regions of more stagnant ice. See [20] for a detailed description of ice sheet dynamics and [25] for a variational approach to ice stream flow.

The common Fluid Mechanics model adopted for cold ice is a non-Newtonian, viscous, heat-conducting, incompressible fluid. It should be pointed out that, strictly speaking, it is not possible to assume ice to be incompressible and yet still presume density variations under phase changes. It is, however, justified to ignore density variations since associated changes in bulk density are very small. On the other hand, it is worth to know that ice sheets are assumed to be isotropic materials, but they can develop an induced anisotropy when stressed over sufficiently long time scales. The model adopted for ice sheet flows result from the basic principles of Fluid Mechanics:

- the conservation of mass

$$\mathrm{div}\mathbf{u} = 0; \tag{4.38}$$

- the conservation of momentum

$$\mathbf{0} = \rho\,\mathbf{g} + \mathrm{div}\mathbf{S}. \tag{4.39}$$

Note that in (4.39) we have neglected the inertial terms by virtue we are in presence of very slow flows. Moreover, we have not written the equation for temperature, which results from the conservation of energy, because in the sequel we will consider only isothermal motions. This brings some controversy to the model and therefore we postpone this assumption for later on. The notation used in (4.38)-(4.39) is well known: $\mathbf{u}$ is the velocity field, ρ is the constant density and $\mathbf{g}$ is the gravitational force. The stress tensor $\mathbf{S}$ and the rate of deformations tensor $\mathbf{D}$ are related by a rheological flow law, denominated as the Glen's law:

$$\mathbf{S} = -p\mathbf{I} + \mathbf{F}(\mathbf{D}), \quad \mathbf{F}(\mathbf{D}) = \eta\, A(\theta)|\mathbf{D}|^{n-1}\mathbf{D}\,, \quad \mathbf{D} = \frac{1}{2}\left(\nabla\,\mathbf{u} + \nabla\,\mathbf{u}\right)^T\,. \tag{4.40}$$

In (4.40), p is the pressure, η is the effective viscosity, $A(\theta)|\mathbf{D}|^{n-1}$ is a cutoff function, $|\mathbf{D}|$ is given by $2|\mathbf{D}| = \mathbf{D} : \mathbf{D}$. Note that $|\mathbf{D}|$ stands for the second stress invariant, D_{II}, of $\mathbf{D}$, because $\mathbf{D} = \mathbf{D}^T$ and (4.38) implies $D_{11} + D_{22} + D_{33} = 0$. A is a temperature-depending rate factor which causes A to vary $\pm 3\,^\circ$K (K=Kelvin) over a temperature range of $50\,^\circ$K. Experimental results showed that n varies from about $1,9$ to $4,8$ in secondary creep and reaches values as high as 10 in tertiary creep. There is general agreement now to use $n = 3$, although Glen concludes that $n = 3,5$ would be more appropriate (see [22, Chapter 2]).

The major simplification of the model ensues by considering the shallow-ice approximation. This is justified, since we assume a physical process in which important length scales in the longitudinal directions are much larger, compared to those in the transverse directions. The shallow-ice approximation consists in the introduction of a stretching transformation in terms of a small parameter $\delta > 0$ for instance $\delta = d/L$, where d is the mean thickness of the ice sheet and L a representative length of the ice sheet. Then the variables and parameters of the governing equations are scaled in terms of δ, which of course is $\ll 1$. A last, but controversy, simplification of our model, results by considering only isothermal ice sheet motions. Although the isothermal models are not quantitatively very realistic, they are mathematically nice. On the other hand, it is not our aim to produce the most realistic model incorporating as much realism as possible. Therefore the omission of the temperature equation is justified and, with no lost of generality, we may assume that $A = 1$ in (4.40). In consequence of all these simplifications, our model reduces to the following equation

$$\frac{\partial h}{\partial t} = \operatorname{div}\left(\frac{(h-l)^{n+2}}{n+2}\left|\nabla h\right|^{n-1}\nabla h + (h-l)\mathbf{u}_b\right) + a\,, \tag{4.41}$$

where $h(x, y, t)$ and $l(x, y)$ are, respectively, the top surface and base of the ice sheet. One can readily see that (4.41) is a nonlinear diffusion equation for the function

$$w(x, y, t) = h(x, y, t) - l(x, y)\,, \tag{4.42}$$

which expresses the local thickness of the ice sheet. The function a expresses the scaled accumulation-ablation rate - the regions where $a > 0$ or $a < 0$ represent, respectively, accumulation or ablation zones. The term $\mathbf{u}_b$ results from assuming the ice sheet slides, with velocity $\mathbf{u}_b$, over its base. This happens when basal ice reaches the melting point and consequently basal melt water is produced. This water can lubricate the bed sufficiently that the ice slides over the bed. But, once the base reaches the melting point, we assume the ice above remains cold. When $\mathbf{u}_b$ is a prescribed function, (4.41) is a nonlinear diffusion-convection equation for w. It corresponds to slip boundary conditions along an assumed temperate bed.

5 Statement of the problem

In this section we introduce the mathematical problem we shall work with. Arguing as in [21, p. 312], we can justify that

$$\mathbf{u}_b \approx -\nabla h\,.$$

Then assuming a ice sheet flow with zero scaled accumulation-ablation rate over a flat base, and using (4.42), we derive from (4.41)

$$\frac{\partial w}{\partial t} + \nabla w \cdot \mathbf{u}_b = \operatorname{div}\left(\frac{w^{n+2}}{n+2}\left|\nabla w\right|^{n-1}\nabla w\right) - 2|\nabla w|^2\,, \tag{5.43}$$

Thus, we consider (5.43) in a cylinder

$$Q_T := \Omega \times (0, T) \subset \mathbb{R}^2 \times \mathbb{R}^+$$

whose boundary is defined by $\Gamma_T := \partial\Omega \times (0, T)$ and where Ω is assumed to be a large enough bounded domain with a sufficiently smooth boundary $\partial\Omega$. We supplement equation (5.43) with its natural initial and boundary conditions:

$$w = h_0 - l \quad \text{in } \Omega \quad \text{when } t = 0\,; \tag{5.44}$$

$$w = 0 \quad \text{on } \Gamma_T\,. \tag{5.45}$$

Note that, in (5.44), $h_0 \equiv h(x, y, 0)$, l is constant and taking the altitude zero reference as the medium sea level, the model itself implies

$$h_0 - l \geq 0\,.$$

It should be remarked that similar mathematical problems to (5.43)-(5.45) have been studied by different authors, among many [3, 15]. In [3] was studied problem (5.43)-(5.45) with a zero basal sliding velocity and were established the finite speed of propagations and waiting time properties for the weak solutions to that problem. It is worth to notice that although the correct mathematical formulation of the problem (5.43)-(5.45) must be done in terms of a parabolic inequality (see [15]), the case in which $a \geq 0$ in Q_T can be treated correctly by replacing the variational inequality by the equation (5.43). We justify this, since it solutions also satisfy the variational inequality formulation and due to the uniqueness result for the variational inequality, they must coincide. We point out that although the results of [15] are valid for the more general cases in which a can be negative in some big region, their results were established for spatially one-dimensional formulations, for $l = 0$ and without the term $|\nabla w|^2$.

Our first new results for the the problem (5.43)-(5.45) concern the study of localization properties of weak solutions such as the finite speed of propagations and the waiting time properties, generalizing the results presented in [3]. We are interested in studying the mathematical problem posed by (5.43)-(5.45) when the basal sliding velocity is given and satisfies to

$$\mathrm{div}\mathbf{u}_b = 0 \quad \text{in} \quad Q_T\,, = \Omega \times (0, T), \tag{5.46}$$

$$\mathbf{u}_b \in \mathrm{C}[0, T; \mathbf{C}^{1+\alpha}(\overline{\Omega})], \quad 0 < \alpha < 1\,. \tag{5.47}$$

6 Localization properties

In this section we shall establish the localization properties of finite speed of propagations and waiting time properties of the weak solutions to the problem (5.43)-(5.45) and if the basal sliding velocity $\mathbf{u}_b$ satisfies (5.46)-(5.47).

Let us consider the non-negative weak solutions of the problem (5.43)-(5.45) which satisfy to

$$\sup_{t\in[0,T]} \int_\Omega |w(\cdot, t)|^2 d\mathbf{x} + \int_0^T \int_\Omega |w|^{n+2} |\nabla w|^{n+1} d\mathbf{x} dt := E(Q_T) < \infty\,. \tag{6.48}$$

Notice that the last estimate follows from the energy relation

$$\frac{d}{2dt}\int_{\Omega}|w(\cdot,t)|^2 d\mathbf{x}+\int_0^T\int_{\Omega}\left(|w|^{n+2}|\nabla w|^{n+1}+w|\nabla w|^2\right)d\mathbf{x}dt=0\,,$$

because $\int_{\Omega}\mathbf{u}_b\nabla w\, d\mathbf{x}=0$ and, according to the Maximum Principle $w\geq 0$. We conjecture that it is possible to prove the existence of such solutions, in a suitable function space, by adapting the arguments used in [9] (see also the reference cited therein to the work of Antontsev, Epikhov and Kashevarov).

In order to define the notions of the properties we want to establish in this part, let us fix $\mathbf{x}_0$ in Ω and assume that

$$w_0(\mathbf{x})=0 \quad \text{for } \mathbf{x}\in B_{\rho_0}(\mathbf{x}_0)=\{\mathbf{x}\in\Omega:|\mathbf{x}-\mathbf{x}_0|<\rho_0\}\subset\Omega\,, \tag{6.49}$$

where $\rho_0\in(0,\operatorname{dist}(\mathbf{x}_0,\partial\Omega))$.

Definition 6.1 *The weak solutions of problem (5.43)-(5.45) possesses the property of:*
1. finite speed of propagation, if for some $\mathbf{x}_0\in\Omega$ *and* $t^*\in(0,T)$

$$w(\mathbf{x},t)=0 \quad a.e.\ in\ B_{\rho(t)}(\mathbf{x}_0) \quad \forall\ t\in[0,t^*]\,;$$

2. waiting time property, if for some $\mathbf{x}_0\in\Omega$ *and* $t^*\in(0,T)$

$$w(\mathbf{x},t)=0 \quad a.e.\ in\ B_{\rho_0}(\mathbf{x}_0) \quad \forall\ t\in[0,t^*]\,.$$

To proceed our study, we consider the Lagrange variables $\mathbf{X}$ defined as usual in Continuum Mechanics (see *e.g.* [24]):

$$\frac{d\mathbf{X}(\mathbf{x},t)}{dt}=\mathbf{u}_b(\mathbf{X},t), \quad t\in(0,T)\,; \tag{6.50}$$

$$\mathbf{X}(\mathbf{x},0)=\mathbf{x}, \quad \mathbf{x}\in\Omega. \tag{6.51}$$

Under conditions (5.46)-(5.47), there exists a unique solution $\mathbf{X}(\mathbf{x},t)$ of problem (6.50)-(6.51), which is a homeomorphism between Ω and $\Omega^t=\{\mathbf{y}:\mathbf{y}=\mathbf{X}(\mathbf{x},t),\ \mathbf{x}\in\Omega\}$ for any $t\in[0,T]$. This solution transforms the ball $B_\rho(\mathbf{x}_0)$ into

$$B^t_\rho(\mathbf{x}_0)=\{\mathbf{y}:\mathbf{y}=\mathbf{X}(\mathbf{x},t),\text{for some }\mathbf{x}\in B_\rho(\mathbf{x}_0)\}.$$

Moreover, the following formula hold

$$\frac{d}{dt}\int_{B^t_\rho(\mathbf{x}_0)}\Phi d\mathbf{y}=\int_{B^t_\rho(\mathbf{x}_0)}\left(\frac{\partial\Phi}{\partial t}+\mathbf{u}_b\nabla\Phi\right)d\mathbf{y}\,, \tag{6.52}$$

$$\frac{dJ}{dt}=J\operatorname{div}\mathbf{u}_b,\ \ J=\det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right),\ \ J(\mathbf{x},0)=\det\left(\frac{\partial\mathbf{X}(\mathbf{x},0)}{\partial\mathbf{x}}\right)=1. \tag{6.53}$$

In the considered case, we have that $J(\mathbf{x},0) = J(\mathbf{x},t) = 1$. We introduce the energy functions

$$
\begin{aligned}
E(\rho,t) &= \int_0^t \int_{B_\rho^t(\mathbf{x}_0)} |w|^{n+2} |\nabla w|^{n+1} d\mathbf{y} d\tau, \\
\frac{\partial E(\rho,t)}{\partial \rho} &= \int_0^t \int_{S_\rho^t(\mathbf{x}_0)} w^{n+3} |\nabla w|^{n-1} \mathbf{n} dS d\tau, \quad S_\rho^t(\mathbf{x}_0) = \partial B_\rho^t(\mathbf{x}_0) \qquad (6.54) \\
B(\rho,t) &= \int_{B_\rho^t(\mathbf{x}_0)} |w|^2 d\mathbf{y}, \quad \overline{B}(\rho,t) = \sup_{0\le s\le t} B(\rho,s).
\end{aligned}
$$

Then, applying the results of [8, Chapter 3], we can prove the following theorem.

Theorem 6.1 *Let w be a non-negative weak solution to the problem (5.43)-(5.45). Assume $\mathbf{u}_b$ satisfies (5.46)-(5.47) and (6.48) holds.*

1. *If (6.49) is verified, then there exists t^*, $0 < t^* < T$, such that*

   $$w(\mathbf{x},t) = 0 \quad a.e. \ in \ B_{\rho(t)}(\mathbf{x}_0), \quad \forall\ t \in [0,t^*],$$

   with $\rho(t)$ given by

   $$\rho^\nu(t) = \rho_0^\nu - \frac{\nu}{\gamma C} t^\lambda E^\gamma(\rho_0, 0),$$

   with some positive constants ν, λ and γ.

2. *If additionally to (6.49), the following condition holds*

   $$\int_{B_\rho(\mathbf{x}_0)} |w_0|^2 d\mathbf{x} \le \varepsilon(\rho-\rho_0)^\mu, \quad \rho > \rho_0, \quad \mu = \mu(n) > 0, \quad \varepsilon > 0,$$

   then, there exist t^*, $0 < t^* < T$, and $\varepsilon^* > 0$, $0 < \varepsilon \le \varepsilon^*$, such that

   $$w(\mathbf{x},t) = 0 \quad a.e. \ in \ B_{\rho_0}(\mathbf{x}_0), \quad \forall\ t \in [0,t^*]\,.$$

PROOF. Multiplying equation (5.43) by w and integrating (formally) over $B_\rho^t(\mathbf{x}_0) \times (0,T)$ with regard to (6.52), (6.53) and (6.54), we come to the energy relation

$$\frac{1}{2} B(\rho,t) + E(\rho,t) = I + \frac{1}{2} B(\rho,0) - \int_0^t \int_{B_\rho^t(\mathbf{x}_0)} w|\nabla w|^2 d\mathbf{y} d\tau, \qquad (6.55)$$

where

$$I = \int_0^t \int_{S_\rho^t(\mathbf{x}_0)} w^{n+3} |\nabla w|^{n-1} \mathbf{n} dS d\tau, \ S_\rho^t(\mathbf{x}_0) = \partial B_\rho^t(\mathbf{x}_0)$$

and $\mathbf{n}$ is the unit outer normal vector to $S^t_\rho(\mathbf{x}_0)$. Notice that $B(\rho,0)=0$ if $\rho \le \rho_0$ which corresponds to the first assertion. In this case, (6.55) leads to the ordinary differential inequality(see [8, §3.2])

$$\left(\overline{B}(\rho,t)+E(\rho,t)\right)^\gamma \le Ct^\lambda \rho^{1-\nu}\left(\frac{\partial E(\rho,t)}{\partial \rho}\right),\ 0<\gamma<1, 0<\lambda,\nu.$$

Integrating last inequality, we come to the estimate

$$E^\gamma(\rho,t)\le E^\gamma(\rho_0,0)-\frac{\gamma}{\nu}Ct^{-\lambda}\left(\rho_0^{1-\nu}-\rho^{1-\nu}\right),$$

which lead us to

$$E(\rho,t)=0,\ \text{if}\ \rho^{1-\nu}\le \rho_0^{1-\nu}-\frac{\nu}{\gamma C}t^\lambda E^\gamma(\rho_0,0).$$

First assertion of the theorem is proved. In the second case, we come to the nonhomogeneous inequality with $\rho \ge \rho_0$

$$\left(\overline{B}(\rho,t)+E(\rho,t)\right)^\gamma \le C\left[t^\lambda\left(\frac{\partial E(\rho,t)}{\partial\rho}\right)+\varepsilon^\gamma(\rho-\rho_0)^{\gamma\mu}\right], \mu\ge \frac{1}{1-\gamma}.$$

According to ([8, §3.3]), all solutions of the last inequality permit the majority

$$E(\rho,t)\le C^\gamma(\rho-\rho_0), \quad \rho\ge\rho_0$$

if $\varepsilon>0$ and $t>0$ are sufficiently small. Second assertion of the theorem is proved.□

Remark 6.1 *It is possible to apply local energy methods, as the presented before and inspired in [8, Chapter 3], for the more general formulation (given in terms of a) of the variational inequality presented in [15], which holds when a can be negative in some big region (always for the case $l=cst$, $\mathbf{u}_b\neq 0$ and $\operatorname{div}\mathbf{u}_b=0$). Indeed, by using parabolic type local energy domains of the type*

$$P(t)\equiv P(t;\vartheta,\upsilon)=\{(x,s)\in Q:|x-x_0|<\rho(s)\equiv\vartheta(s-t)^\upsilon, s\in(t,T)\}$$

for suitable choices of the parameters ϑ and υ ($\vartheta>0$, $0<\upsilon<1$), and by defining the local energy functions

$$E(P):=\int_{X(t,P(t))}|w|^{n+2}|\nabla w|^{n+1}d\mathbf{y}d\tau,\quad C(P):=\int_{X(t,P(t))}|w(\mathbf{y},\tau)|\,d\mathbf{y}d\tau$$

$$b(T):=ess\sup_{s\in(t,T)}\int_{X(t,P(t))}|w|^2d\mathbf{y},$$

we can adapt the results of [8, §3.4] in a similar way as was done for the Stefan and obstacle problems in [16] to prove dead core type properties. I.e. *even if we assume initially that $h(\mathbf{x},0)=h_0(\mathbf{x})>l$ for a.e. $\mathbf{x}\in\Omega$, if $h(\mathbf{x},t)<-\varepsilon<0$ a.e. on $B_\rho(\mathbf{x}_0)\subset\Omega$, $t\in(0,T)$, for some $\varepsilon>0$ and some $B_\rho(\mathbf{x}_0)$, then $h(\mathbf{x},t)=0$ on a positive measured subset of the form $P(t)$, for t large enough.*

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