Higher central extensions and cohomology

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Always in this talk: $Z$ an object, $A$ an abelian object

\[
\begin{align*}
\text{degree 1} & \quad H^2(Z, A) \cong \text{Centr}^1(Z, A) \\
& \quad \text{classical for groups: } 0 \rightarrow A \xrightarrow{\cdot f} X \xrightarrow{f} Z \rightarrow 0 \quad f \text{ central extension: regular epimorphism with } [A, X] = 0 \\
& \quad \text{semi-abelian monadic case: } [\text{Gran–VdL, 2008}] \\
\text{degree 2} & \quad H^3(Z, A) \cong \text{Centr}^2(Z, A) \\
& \quad [\text{Rodelo–VdL, 2010}] \text{ based on } [\text{Everaert–Gran–VdL, 2008}] \text{ and G. Janelidze’s work on categorical Galois theory} \\
& \quad \text{left: cohomology “without projectives” of } [\text{Bourn 1999, 2002}] \text{ and } [\text{Bourn–Rodelo, 2007}], \text{ notion of direction} \\
& \quad \text{right: classes of double central extensions of } Z \text{ by } A
\end{align*}
\]
Cohomology and central extensions

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**degree 1** \( H^2(Z, A) \cong \text{Centr}^1(Z, A) \)

- classical for groups: \( 0 \to A \to X \stackrel{f}{\to} Z \to 0 \)
  \( f \) central extension: regular epimorphism with \( [A, X] = 0 \)
- semi-abelian monadic case: [Gran–VdL, 2008]

**degree 2** \( H^3(Z, A) \cong \text{Centr}^2(Z, A) \)

- right: classes of double central extensions of \( Z \) by \( A \)
Cohomology and central extensions

degrees $n \geq 2$ \quad $H^{n+1}(Z, A) \cong \text{Centr}^n(Z, A)$

- the subject of this talk, recent work of Rodelo–VdL
  - first algebraic proof for groups, now general proof which is geometric
  - left: cohomology as classes of higher torsors [Duskin 1975, 1979] and [Glenn, 1982]
    in the monadic case, Barr–Beck comonadic cohomology
- right: classes of higher central extensions
- framework: semi-abelian categories + (CC)
Cohomology and central extensions

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Higher central extensions

\( \mathcal{A} \) semi-abelian category; \( 0 = \emptyset \) and \( n + 1 = \{0, \ldots, n\} \)

Cubes and extensions

- an \textbf{n-cube} in \( \mathcal{A} \) is a functor \( F: (2^n)^{\text{op}} \to \mathcal{A} \)
- an \textbf{n-cube} \( F \) is an \textbf{n-extension} iff for all \( \emptyset \neq I \subseteq n \)
  \( F_I \to \lim_{J \subseteq I} F_J \) is regular epi

Inductive definition (Galois theory, after [Janelidze–Kelly, 1994])

- \( \text{Ab}\mathcal{A} \subset \mathcal{A} \) full reflective subcategory
- \( \text{CExt}^1\mathcal{A} \subset \text{Ext}^1\mathcal{A} \): central w.r.t. \( \text{Ab}\mathcal{A} \)
- \( \text{CExt}^2\mathcal{A} \subset \text{Ext}^2\mathcal{A} \): central w.r.t. \( \text{CExt}^1\mathcal{A} \)
- \( \text{CExt}^{n+1}\mathcal{A} \subset \text{Ext}^{n+1}\mathcal{A} \): central w.r.t. \( \text{CExt}^n\mathcal{A} \)

Gives adjunctions \( \text{CExt}^n\mathcal{A} \overset{\subset}{\underset{I_n}{\leftarrow}} \text{Ext}^n\mathcal{A} \)
The direction of a three-fold (central) extension
Higher central extensions


Take an object $Z$ of $\mathcal{A}$ and $n \geq 1$. For any $n$-presentation $F$ of $Z$,

$$H_{n+1}(Z, \text{Ab}\mathcal{A}) \cong \frac{\langle F_n \rangle \cap \bigcap_{i \in n} K[f_i]}{L_n[F]}$$

- $F_n$ initial object of the cube, the $f_i$ the initial arrows
- exact sequence $0 \to \langle X \rangle \xrightarrow{} X \xrightarrow{\eta_X} \text{ab}X \xrightarrow{} 0$ for any $X$ so $\langle X \rangle = [X, X]$, the Huq commutator
- an $n$-extension $F$ is central iff $L_n[F] = 0$
- $\bigcap_{i \in n} K[f_i] = K^n[F] = D_{(n,Z)}F$ is the direction of $F$,

$$D_{(n,Z)}: \text{CExt}^n_{Z,\mathcal{A}} \to \text{Ab}\mathcal{A}: F \mapsto D_{(n,Z)}F = \bigcap_{i \in n} K[f_i]$$

- $H_{n+1}(Z, \text{Ab}\mathcal{A}) \cong \lim D_{(n,Z)}$ by [Goedecke–VdL, 2009]
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- $H_{n+1}(Z, \text{AbA}) \cong \text{lim } D_{(n,Z)}$ by [Goedecke–VdL, 2009]
The commutator condition (CC)

**Definition**

A semi-abelian category satisfies the **commutator condition (CC)** when for all \( n \geq 1 \), an \( n \)-fold extension \( F \) is central iff

\[
\left[ \bigcap_{i \in I} K[f_i], \bigcap_{i \in n \setminus I} K[f_i] \right] = 0
\]

for all \( I \subseteq n \). (Hence \( L_n[F] = \bigcup_{I \subseteq n} \left[ \bigcap_{i \in I} K[f_i], \bigcap_{i \in n \setminus I} K[f_i] \right] \).)

- In degree 1, all semi-abelian categories satisfy (CC)
- in degree 2, (CC) is weaker than (SH) “Smith = Huq” by [Rodelo–VdL, 2010]
- so far, in degrees \( n \geq 3 \), we only have examples: groups, non-unitary rings, Lie algebras, etc., besides all semi-abelian arithmetical and all abelian categories
- Is (CC) a higher-dimensional version of (SH)?
Main theorem, consequences

**Theorem**

In a semi-abelian category with (CC), let $Z$ be an object and $A$ an abelian object. Consider $n \geq 1$. Then

$$H^{n+1}(Z, A) \cong \text{Centr}^n(Z, A) = \pi_0(D_{(n,Z)}^{-1}A)$$

where $H^{n+1}(Z, A)$ is Duskin–Glenn cohomology, and Barr–Beck comonadic cohomology in the monadic case; $\text{Centr}^n(Z, A)$ contains equivalence classes of **central extensions of** $Z$ by $A$.

- Long exact sequence for $\text{Centr}^n(Z, -)$
- Duality in the *interpretations* of homology and cohomology:

$$H_{n+1}(Z, \text{Ab}\mathcal{A}) \cong \lim D_{(n,Z)}^{-1} \quad H^{n+1}(Z, A) \cong \pi_0(D_{(n,Z)}^{-1}A)$$

where $D_{(n,Z)} : \text{CExt}_Z^n\mathcal{A} \to \text{Ab}\mathcal{A} : F \mapsto D_{(n,Z)}F = \bigcap_{i \in n} K[f_i]$
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The direction of a three-fold (central) extension
Duskin and Glenn’s torsors:
A “simplicial” version of higher central extensions

\[
\begin{array}{ccc}
\text{torsor} & = & \text{truncated simplicial resolution} \\
\text{central extension} & = & \text{extension} \\
& = & \text{pregroupoid}
\end{array}
\]

- Groupoid
  - \( G_1 \)
  - \( G_0 \)
  - Multiplication, identities
  - Only one object of objects

- Pregroupoid
  - \( G_1 \)
  - \( G_0 \)
  - \( G_0' \)
  - Mal’tsev operation
  - Two objects of objects

\[
m(\alpha, \beta) = \gamma
\]

\[
p(\alpha, \beta, \gamma) = \delta
\]
Duskin and Glenn’s torsors:
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torsor
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= truncated simplicial resolution
extension
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pregroupoid

groupoid

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G_1 \\
\sigma_0 \quad \sigma_1 \\
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Duskin and Glenn’s torsors: Definition

- Let $Z$ be an object, $A$ an abelian object
- $\mathcal{K}(Z, A, n)$ is the augmented simplicial object

\[
\begin{array}{cccccc}
& & n+1 & n & n-1 & n-2 \\
A^{n+1} \times Z & \overset{\partial_{n+1} \times 1_Z}{\rightarrow} & A \times Z & \overset{\text{pr}_Z}{\rightarrow} & Z & \cdots \\
& \overset{\text{pr}_n \times 1_Z}{\rightarrow} & & \overset{\text{pr}_Z}{\rightarrow} & & \\
& \overset{\text{pr}_0 \times 1_Z}{\rightarrow} & & & & \\
\end{array}
\]

where $\partial_{n+1} = (-1)^n \sum_{i=0}^n (-1)^i \text{pr}_i$

- an $n$-torsor of $Z$ by $A$ is an augmented simplicial object $\mathcal{T}$ together with a morphism $\xi: \mathcal{T} \rightarrow \mathcal{K}(Z, A, n)$ such that
  - $(T1)$ $\xi$ is a fibration, exact from degree $n$ on;
  - $(T2)$ $\mathcal{T} \cong \text{Cosk}_{n-1} \mathcal{T}$;
  - $(T3)$ $\mathcal{T}$ is a simplicial resolution

- $(T1)$ means $\Delta(\mathcal{T}, n) \cong A \times \wedge^i(\mathcal{T}, n)$ for all $i$; in particular $A \cong \bigcap_{i \in n} K[\partial_i]$
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  & A \times Z & \xrightarrow{\partial_i} & Z \\
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  & \quad \downarrow{\text{pr}_0 \times 1_Z} & & \quad \downarrow{\text{pr}_Z} \\
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Duskin and Glenn’s torsors: Fundamental results

**Definition/Theorem (Duskin–Glenn)**

\[ H^{n+1}(Z, A) \cong \pi_0 Tors^n(Z, A) \] where \( Tors^n(Z, A) \) is considered as a full subcategory of \( S^+A/\mathbb{K}(Z, A, n) \)

**Theorem**

A simplicial object is an \( n \)-torsor iff its \((n - 1)\)-truncation is an \( n \)-fold central extension

\[ \Rightarrow \] depends on (CC), algebraic proof

\[ \Leftarrow \] always true, uses *geometry of higher central extensions*

**Proposition**

Every central extension is connected with a central truncated simplicial resolution: every class of \( D_{(n, Z)}^{-1}A \) contains a torsor of \( Z \) by \( A \)
**Duskin and Glenn’s torsors:**

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The geometry of higher central extensions in degree 2: box operation, diamonds

\[
\begin{align*}
X \xrightarrow{c} C \\
F \downarrow \\
D \xrightarrow{d} Z
\end{align*}
\]

is central iff \( R[d] \Box R[c] \cong A \times (R[d] \times \chi R[c]) \)

\( R[d] \Box R[c] \) contains diamonds

\( R[d] \times \chi R[c] \) diamonds with one arrow missing

notation \( R[d] \times \chi R[c] = R[d] \Box^0 R[c] \)
Higher-order box operation: \( \Box_i R[f_i] \) in degree 3
The elements of $\square_i R[f_i]$ in degree 3

- in degree 3, the diamonds are octahedra, represented by matrices of order $2 \times 2 \times 2 = 2^3$ via geometric duality:

- in $\square_i^3 R[f_i]$ the triangle $b$ is missing, since $3 = \{0, 1, 2\}$
- if $F$ is central, this triangle is (uniquely) determined by an element of the direction $A$, as $\square_i R[f_i] \cong A \times \square_i^3 R[f_i]$
- any cycle may be embedded into a diamond
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The elements of $\bigotimes_i R[f_i]$ in degree 3

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- any cycle may be embedded into a diamond
In a semi-abelian category \( \mathcal{A} \) which satisfies (CC)

### Correspondence between torsors and central extensions

\[
H^{n+1}(Z, A) \cong \pi_0 \text{Tors}^n(Z, A) \cong \text{Centr}^n(Z, A)
\]

### Duality between homology and cohomology

\[
D_{(n, Z)} : \text{CExt}^n_{\mathbb{Z}} \mathcal{A} \to \text{Ab} \mathcal{A} : F \mapsto D_{(n, Z)} F = \bigcap_{i \in n} K[f_i]
\]

\[
H_{n+1}(Z, \text{Ab} \mathcal{A}) \cong \lim D_{(n, Z)} \quad H^{n+1}(Z, A) \cong \pi_0(D^{-1}_{(n, Z)} A)
\]

### To do

Extend to non-trivial coefficients
Characterise the commutator condition in elementary terms